New Developments in Projective Geometric Algebra

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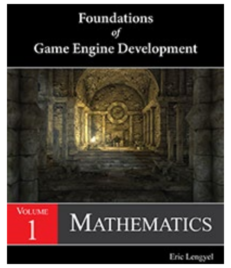
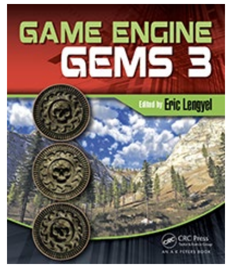
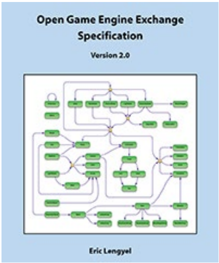
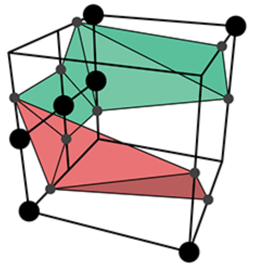
About the Speaker

- Working in game/graphics dev since 1994
  - Previously at Sierra, Apple, Naughty Dog
- Current projects:
  - Slug Library, C4 Engine, The 31st, FGED, OpenGEX









More Information

- projectivegeometricalgebra.org
- Past GDC sessions on Grassmann algebra
- Foundations of Game Engine Development, Volume 1: Mathematics

Outline

- Take a look at conventional math
  - Pieces of a puzzle, but big picture missing
- Review of Grassmann algebra
  - With some new stuff added
- New developments in geometric algebra
  - Antiproducts, geometric norms, motors, flectors

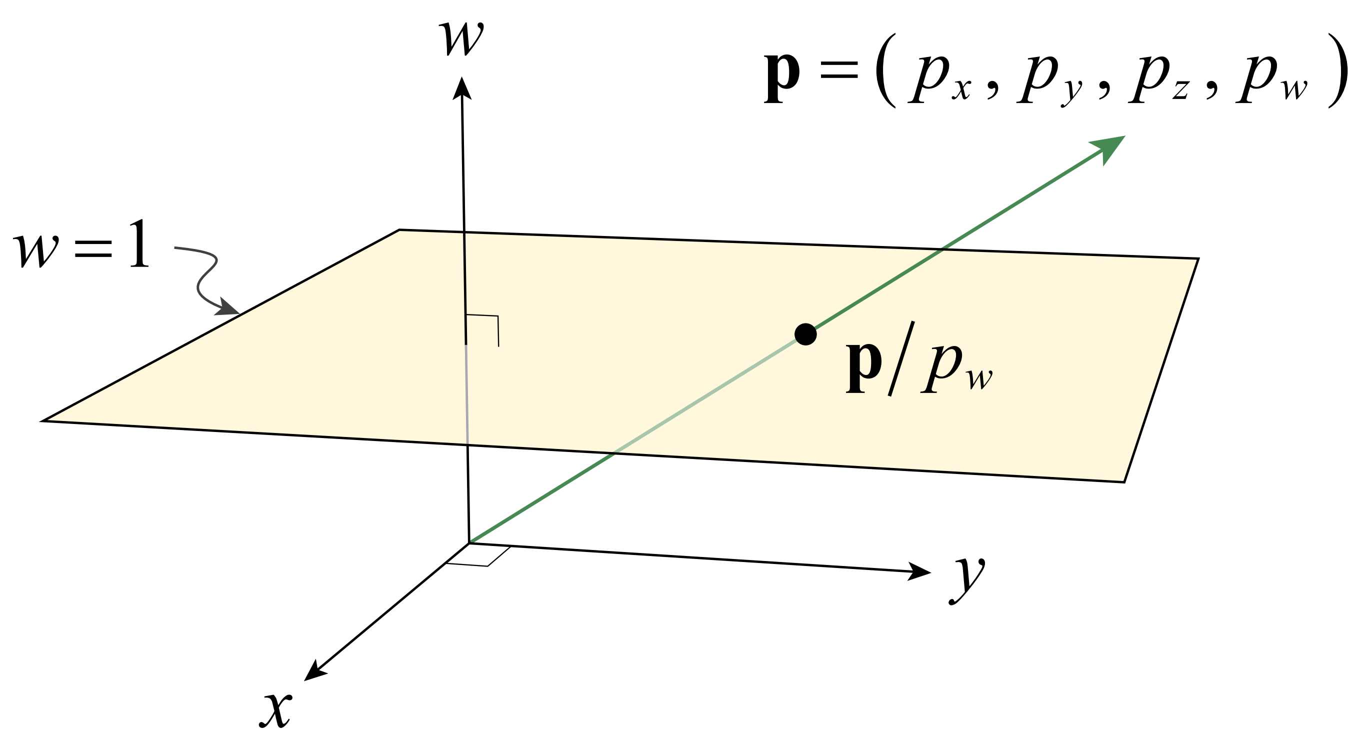
Homogeneous Coordinates

- Add *w* coordinate to make 4D vector
- Points have $w \neq 0$
- Directions have $w = 0$
- Allows rotation and translation to be combined in a single 4×4 matrix:

$$\mathbf{p}' = \begin{bmatrix} R_{00} & R_{01} & R_{02} & T_x \\ R_{10} & R_{11} & R_{12} & T_y \\ R_{20} & R_{21} & R_{22} & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ p_w \end{bmatrix}$$

Homogeneous Coordinates

- "Homogeneous" means any scalar multiple of a vector has the same geometric meaning
- Project into 3D space by intersecting with the plane $w = 1$



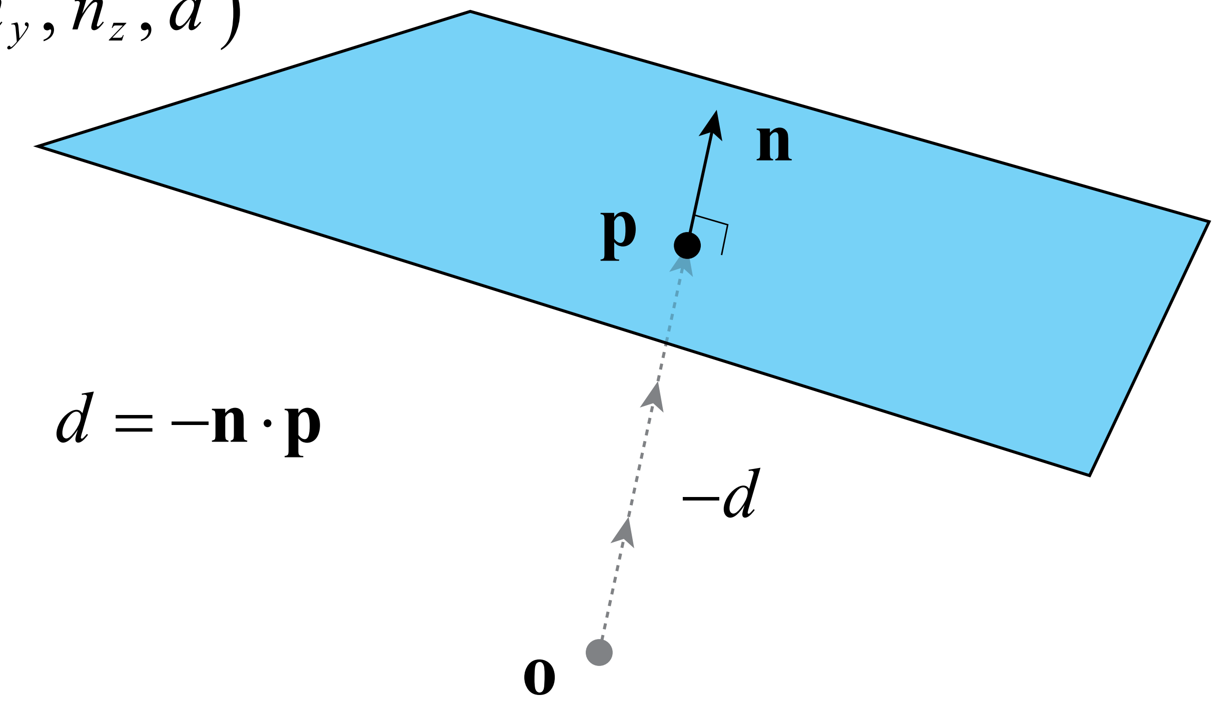
Implicit Planes

- Four-component quantity (n_x, n_y, n_z, d)
- $\mathbf{n}$ is normal vector
- d is signed distance from origin, scaled by length of $\mathbf{n}$

- Planes are also homogeneous
  - Any scalar multiple is same plane

Implicit Planes

$$\mathbf{f} = \left(n_x, n_y, n_z, d\right)$$



$$d = -\mathbf{n} \cdot \mathbf{p}$$

Plücker Coordinates

- Parametric form of line:

$$\mathbf{L}(t) = \mathbf{p} + t\mathbf{v}$$

- Plücker coordinates give implicit line:

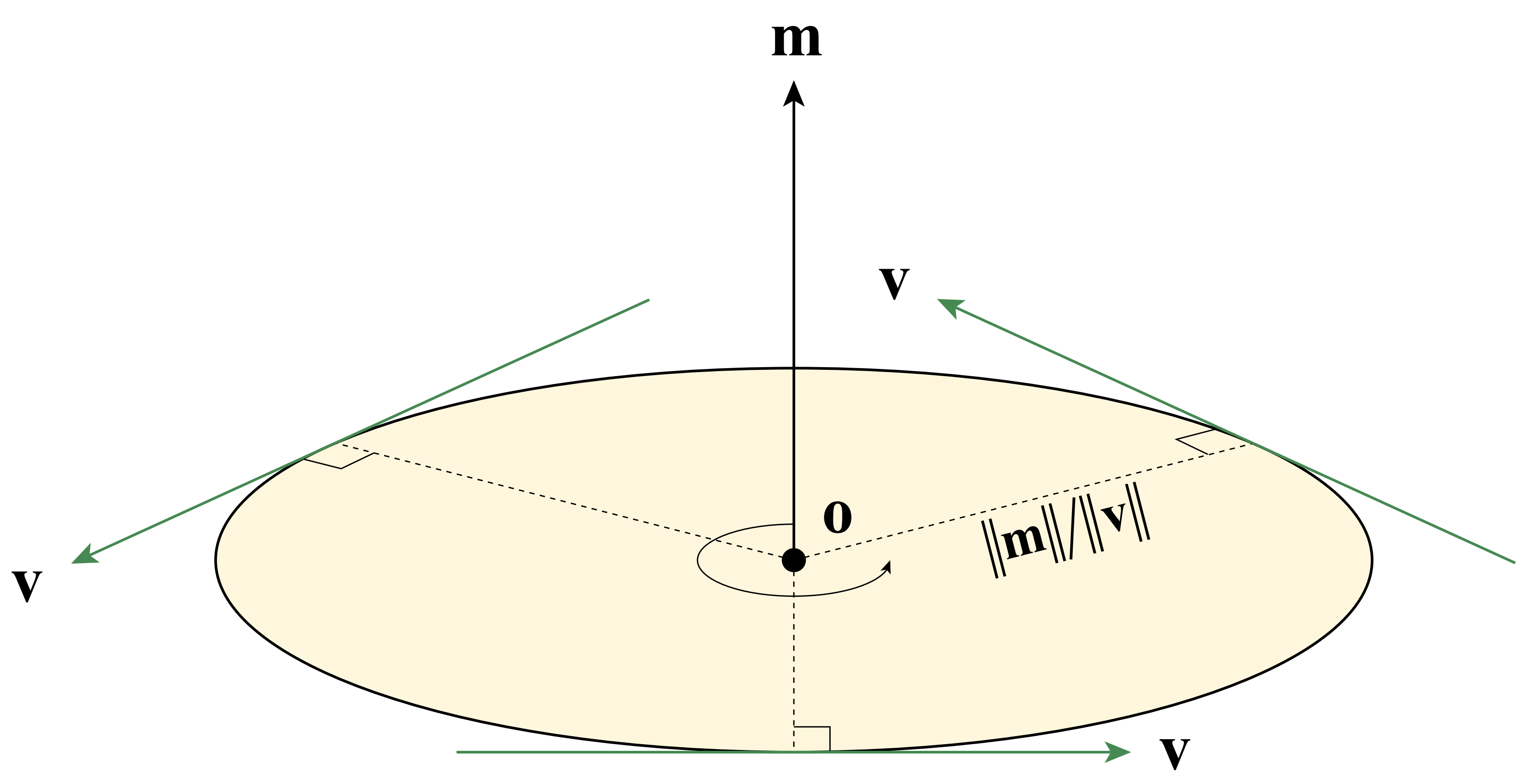
$$\mathbf{v} = \mathbf{q} - \mathbf{p}$$

$$\mathbf{m} = \mathbf{p} \times \mathbf{q}$$

Plücker Coordinates

- $\mathbf{v}$ is the direction of the line
- $\mathbf{m}$ is the moment of the line
- Always true that $\mathbf{v} \cdot \mathbf{m} = 0$
- Representation contains no information about the points used to create the line

Direction and Moment



Plücker Coordinates

- Lots of formulas
  - But little explanation

- Point $(\mathbf{p} \mid w)$
- Plane $[\mathbf{n} \mid d]$
- Line $\{\mathbf{v} \mid \mathbf{m}\}$

	Formula	Description
A	$\{\mathbf{v} \mid \mathbf{p} \times \mathbf{v}\}$	Line through point **p** with direction **v**.
B	$\{\mathbf{p}_2 - \mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two points $\mathbf{p}_1$ and $\mathbf{p}_2$.
C	$\{\mathbf{p} \mid \mathbf{0}\}$	Line through point **p** and origin.
D	$\{w_1\mathbf{p}_2 - w_2\mathbf{p}_1 \mid \mathbf{p}_1 \times \mathbf{p}_2\}$	Line through two homogeneous points $(\mathbf{p}_1 \mid w_1)$ and $(\mathbf{p}_2 \mid w_2)$.
E	$\{\mathbf{n}_1 \times \mathbf{n}_2 \mid d_1\mathbf{n}_2 - d_2\mathbf{n}_1\}$	Line where two planes $[\mathbf{n}_1 \mid d_1]$ and $[\mathbf{n}_2 \mid d_2]$ intersect.
F	$(\mathbf{m} \times \mathbf{n} + d\,\mathbf{v} \mid -\mathbf{n} \cdot \mathbf{v})$	Homogeneous point where line $\{\mathbf{v} \mid \mathbf{m}\}$ intersects plane $[\mathbf{n} \mid d]$.
G	$(\mathbf{v} \times \mathbf{m} \mid v^2)$	Homogeneous point closest to origin on line $\{\mathbf{v} \mid \mathbf{m}\}$.
H	$(-d\mathbf{n} \mid n^2)$	Homogeneous point closest to origin on plane $[\mathbf{n} \mid d]$.
I	$[\mathbf{v} \times \mathbf{u} \mid -\mathbf{u} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and parallel to direction **u**.
J	$[\mathbf{v} \times \mathbf{p} + \mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and point **p**.
K	$[\mathbf{m} \mid 0]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and origin.
L	$[\mathbf{v} \times \mathbf{p} + w\mathbf{m} \mid -\mathbf{p} \cdot \mathbf{m}]$	Plane containing line $\{\mathbf{v} \mid \mathbf{m}\}$ and homogeneous point $(\mathbf{p} \mid w)$.
M	$[\mathbf{m} \times \mathbf{v} \mid m^2]$	Plane farthest from origin containing line $\{\mathbf{v} \mid \mathbf{m}\}$.
N	$[-w\mathbf{p} \mid p^2]$	Plane farthest from origin containing point $(\mathbf{p} \mid w)$.
O	$\frac{\lvert \mathbf{v}_1 \cdot \mathbf{m}_2 + \mathbf{v}_2 \cdot \mathbf{m}_1 \rvert}{\lVert \mathbf{v}_1 \times \mathbf{v}_2 \rVert}$	Distance between two lines $\{\mathbf{v}_1 \mid \mathbf{m}_1\}$ and $\{\mathbf{v}_2 \mid \mathbf{m}_2\}$.
P	$\frac{\lVert \mathbf{v} \times \mathbf{p} + \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to point **p**.
Q	$\frac{\lVert \mathbf{m} \rVert}{\lVert \mathbf{v} \rVert}$	Distance from line $\{\mathbf{v} \mid \mathbf{m}\}$ to origin.
R	$\frac{\lvert \mathbf{n} \cdot \mathbf{p} + d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to point **p**.
S	$\frac{\lvert d \rvert}{\lVert \mathbf{n} \rVert}$	Distance from plane $[\mathbf{n} \mid d]$ to origin.

Table from *Foundations of Game Engine Development, Volume 1: Mathematics*, Section 3.5.2.

Quaternions

- Encodes arbitrary rotation about origin:

$$\mathbf{q} = \cos\phi + \mathbf{a}\sin\phi$$

- This is a rotation through the angle 2ϕ about the unit-length axis $\mathbf{a}$.

Quaternions

- Quaternions often written as

$$\mathbf{q} = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

- Conjugate negates “imaginary” parts:

$$\tilde{\mathbf{q}} = w - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$$

Quaternions

- Units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ multiply as follows:

$$\mathbf{i}^2 = -1 \qquad \mathbf{ij} = \mathbf{k}$$

$$\mathbf{j}^2 = -1 \qquad \mathbf{jk} = \mathbf{i}$$

$$\mathbf{k}^2 = -1 \qquad \mathbf{ki} = \mathbf{j}$$

Quaternions

- A vector $\mathbf{v}$ is rotated by the sandwich product:

$$\mathbf{v}' = \mathbf{q}\mathbf{v}\tilde{\mathbf{q}}$$

- where $\mathbf{v}$ is regarded as the quaternion

$$\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$$

Dual Quaternions

- Quaternions can rotate only about the origin
- They cannot handle translations
- Just like a 3×3 matrix

- Dual quaternions incorporate translations
- This also allows rotation about arbitrary lines
- Analogous to 4×4 matrices

Dual Quaternions

- Dual quaternion conventionally written as a pair of quaternions:

$$\mathbf{q}_r + \varepsilon \mathbf{q}_d$$

- $\mathbf{q}_r$ is the real part
- $\mathbf{q}_d$ is the dual part
- ε squares to zero: $\varepsilon^2 = 0$

Dual Quaternions

- A point $\mathbf{p}$ is transformed by a dual quaternion by first writing $\mathbf{p}$ as

$$\mathbf{p} = 1 + \varepsilon \mathbf{i} p_x + \varepsilon \mathbf{j} p_y + \varepsilon \mathbf{k} p_z$$

- Then, the sandwich product is applied:

$$\mathbf{p}' = (\mathbf{q}_{\mathrm{r}} + \varepsilon \mathbf{q}_{\mathrm{d}})(1 + \varepsilon \mathbf{i} p_x + \varepsilon \mathbf{j} p_y + \varepsilon \mathbf{k} p_z)(\tilde{\mathbf{q}}_{\mathrm{r}} + \varepsilon \tilde{\mathbf{q}}_{\mathrm{d}})$$

Hacks!

- The dual quaternion transformation technique is an ugly hack
  - We will see that points are being cast to translation operators, transformed, and then cast back to points

- Quaternion rotations are a lesser hack, but still a hack
  - Vectors are being cast to bivectors

Hacks!

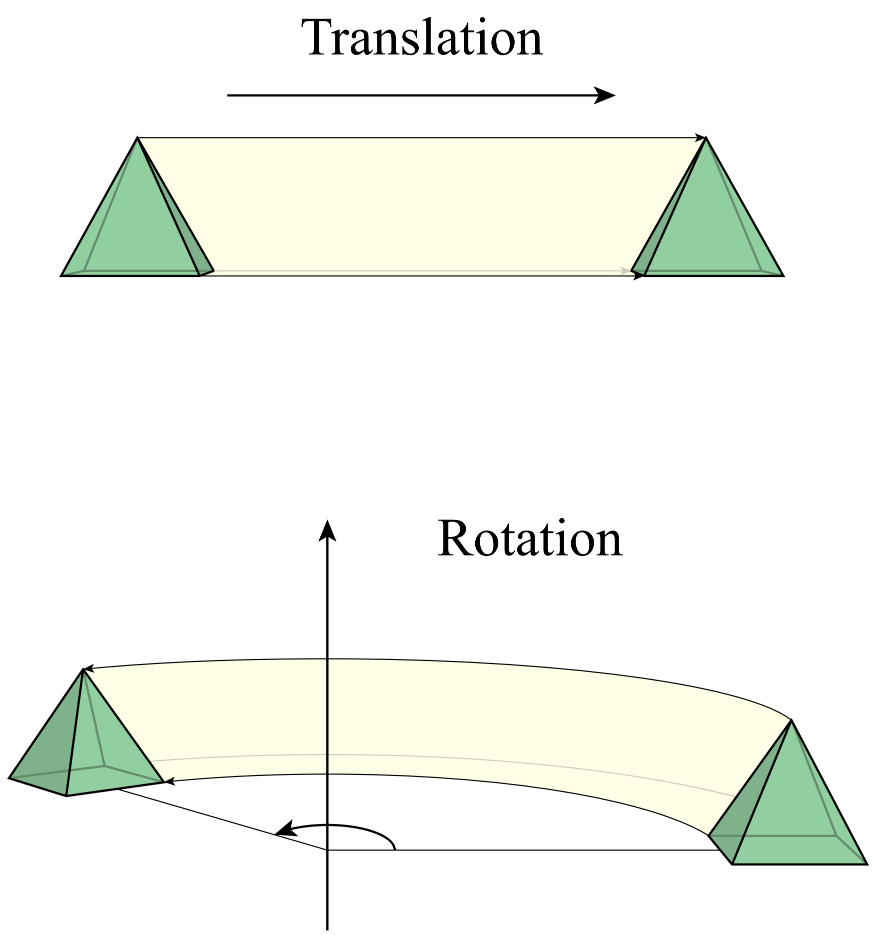
- Conventional dual quaternion methods do not handle other types of objects
  - Like lines and planes

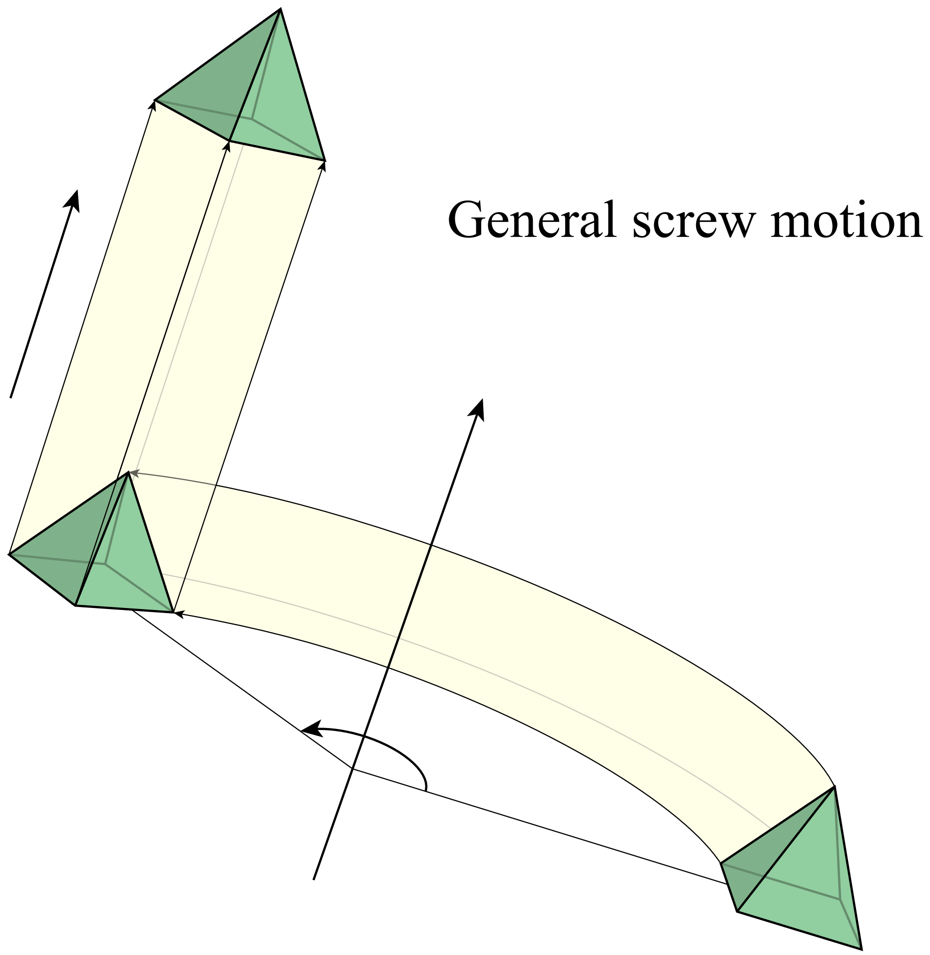
- We are going to fix this and fill in some giant holes in the theory

What About Reflections?

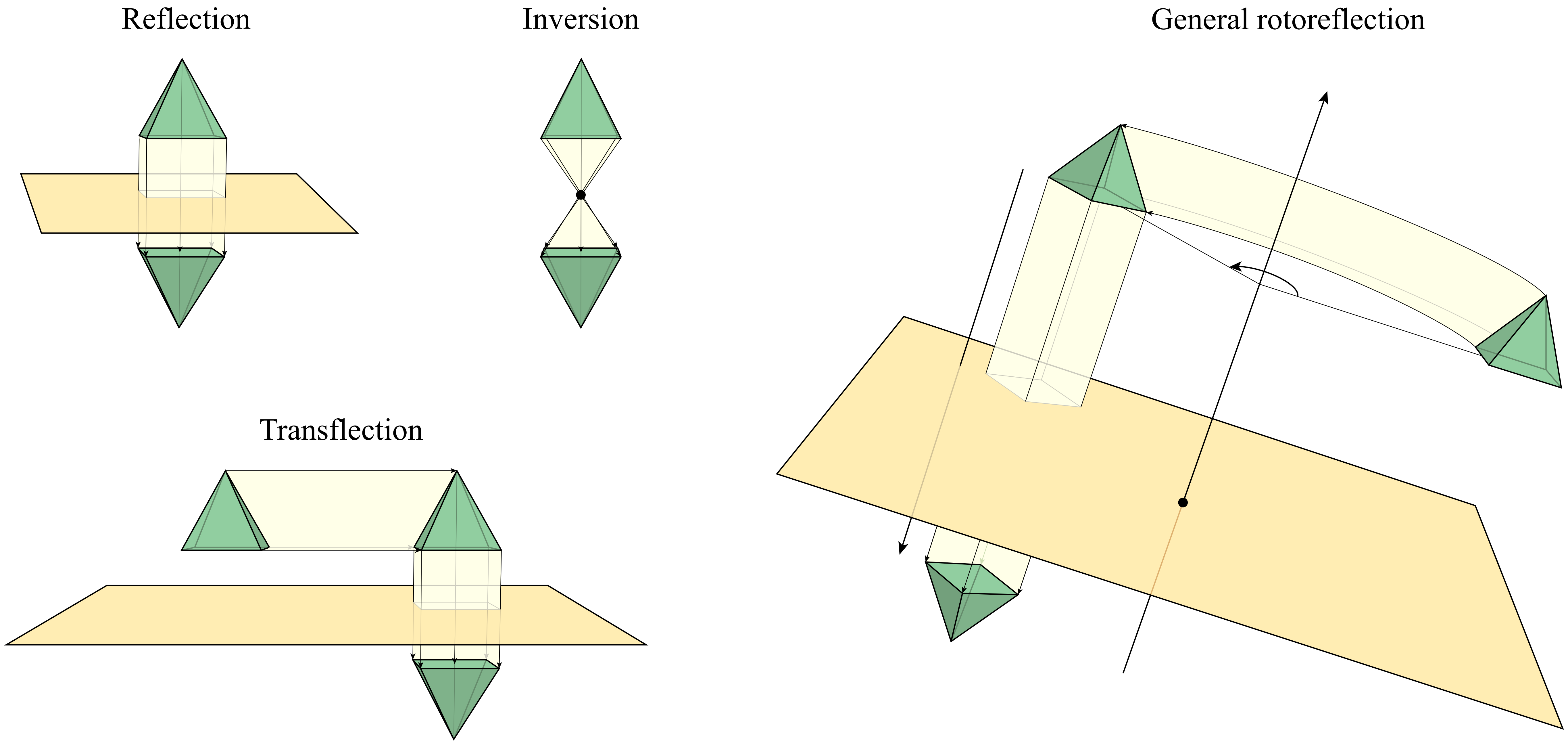
- Dual quaternions give us rotations and translations
- The full set of Euclidean isometries includes improper transformations
  - Reflections
  - Inversions
  - Transflections
  - Rotoreflections

Proper Euclidean Isometries





Improper Euclidean Isometries



Projective Geometric Algebra (PGA)

- A four-dimensional projective space
- Point, line, and plane representations
  - With operations for combining in various ways
- Natural operations for all Euclidean isometries
  - Works with everything in the algebra
  - Both proper and improper transformations

Grassmann Algebra

- Also called exterior algebra
- Contains everything in the geometric algebra

- Fundamental geometric operations
  - Combine geometries with join and meet operations
  - Perform projection of one geometry onto another

- Isometries are part of full geometric algebra

Wedge Product

- Also known as exterior product
  - Grassmann called it progressive combinatorial product

- Written with upward wedge:

$$\mathbf{a} \wedge \mathbf{b}$$

- Read as “$\mathbf{a}$ wedge $\mathbf{b}$”

Wedge Product

- The square of a vector is always zero: $\mathbf{v} \wedge \mathbf{v} = 0$

- This implies that vectors anticommute:

$$(\mathbf{a}+\mathbf{b}) \wedge (\mathbf{a}+\mathbf{b}) = 0$$

$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{a} = 0$$

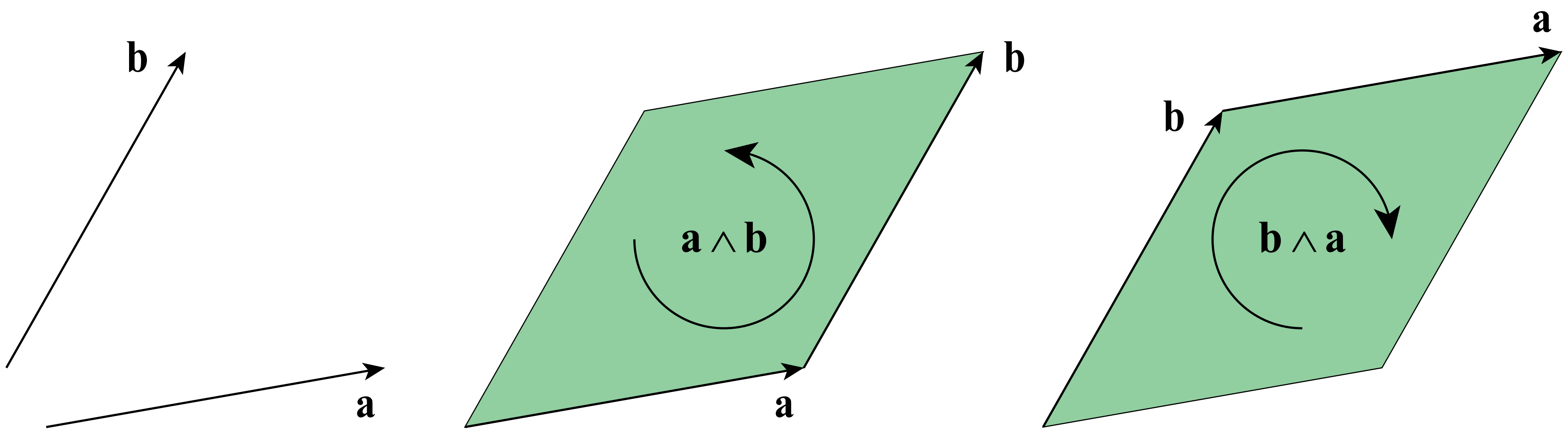
$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Bivectors

- Wedge product of two vectors is a “bivector”
  - Distinct from scalar or vector
  - Represents an oriented 2D area
    - Whereas a vector represents an oriented 1D direction

Bivectors

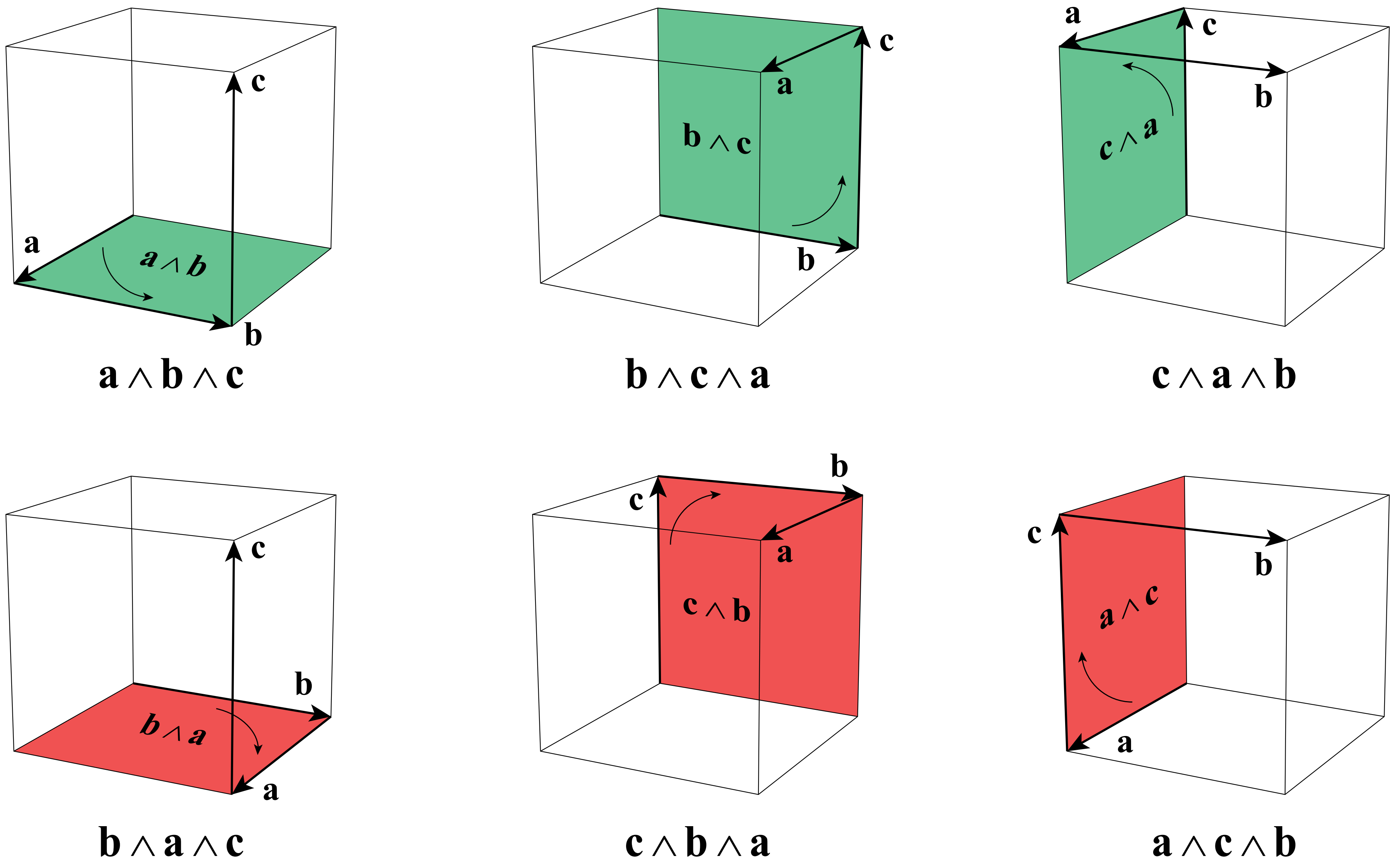
- A bivector is two directions and a magnitude



Trivectors

- Wedge product of three vectors is a “trivector”
  - Another distinct type of object
  - Represents an oriented 3D volume
  - Three directions and a magnitude

Trivectors



a
b
c
$a \wedge b$
$a \wedge b \wedge c$
a
b
c
$b \wedge c$
$b \wedge c \wedge a$
a
b
c
$c \wedge a$
$c \wedge a \wedge b$
a
b
c
$b \wedge a$
$b \wedge a \wedge c$
a
b
c
$c \wedge b$
$c \wedge b \wedge a$
a
b
c
$a \wedge c$
$a \wedge c \wedge b$

Basis Elements in 4D Space

- Four basis vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
- Six basis bivectors: $\mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}$
- Four basis trivectors: $\mathbf{e}_{234}, \mathbf{e}_{314}, \mathbf{e}_{124}, \mathbf{e}_{321}$

Antivectors

- Vectors use basis elements having one dimension each:

$$x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + w\mathbf{e}_4$$

- Antivectors use basis elements having all except one dimension each:

$$x\mathbf{e}_{234} + y\mathbf{e}_{314} + z\mathbf{e}_{124} + w\mathbf{e}_{321}$$

Scalars and Antiscalars

- There are two subspaces of single-component quantities, called scalars and antiscalars
- Scalars include no dimensions of space
- Antiscalars include all dimensions of space

Scalars and Antiscalars

- We represent the scalar basis element by a bold number one: $\mathbf{1}$

- We represent the antiscalar basis element by a blackboard bold number one: 1

$$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$$

- “Anti-one”

Grade and Antigrade

- The grade of an element is the number of dimensions used by its components

- The antigrade of an element is the number of dimensions **not** used by its components

- These, of course, always sum to the total dimension of the algebra

Basis Elements

Type	Basis Elements	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$ $\mathbf{e}_4$	1 / 3	▮▯▯▯ ▯▮▯▯ ▯▯▮▯ ▯▯▯▮
Bivectors	$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$ $\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$ $\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$ $\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$ $\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$ $\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$	2 / 2	▯▮▮▯ ▮▯▮▯ ▮▮▯▯ ▯▯▮▮ ▯▮▯▮ ▮▯▯▮
Trivectors / Antivectors	$\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$ $\mathbf{e}_{124} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$ $\mathbf{e}_{314} = \mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4$ $\mathbf{e}_{234} = \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	3 / 1	▮▮▮▯ ▮▮▯▮ ▮▯▮▮ ▯▮▮▮
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

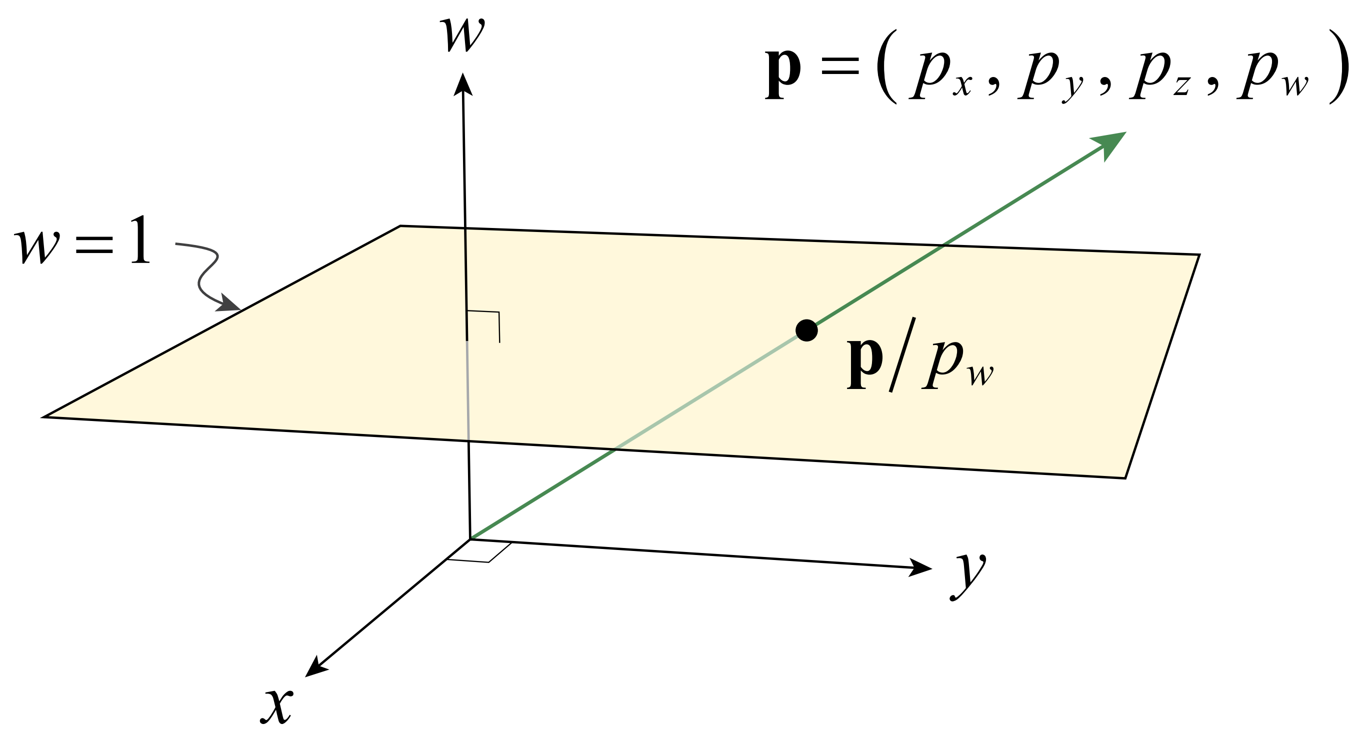
Homogeneous Point

- Ordinary vector

$$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$$

Homogeneous Point

- Projection of 1D vector into subspace at $w = 1$ is a 0D point



Point at Infinity

- If *w* coordinate is zero, then vector represents a point at infinity in the (x, y, z) direction

- Each point at infinity exists in one direction

Homogeneous Line

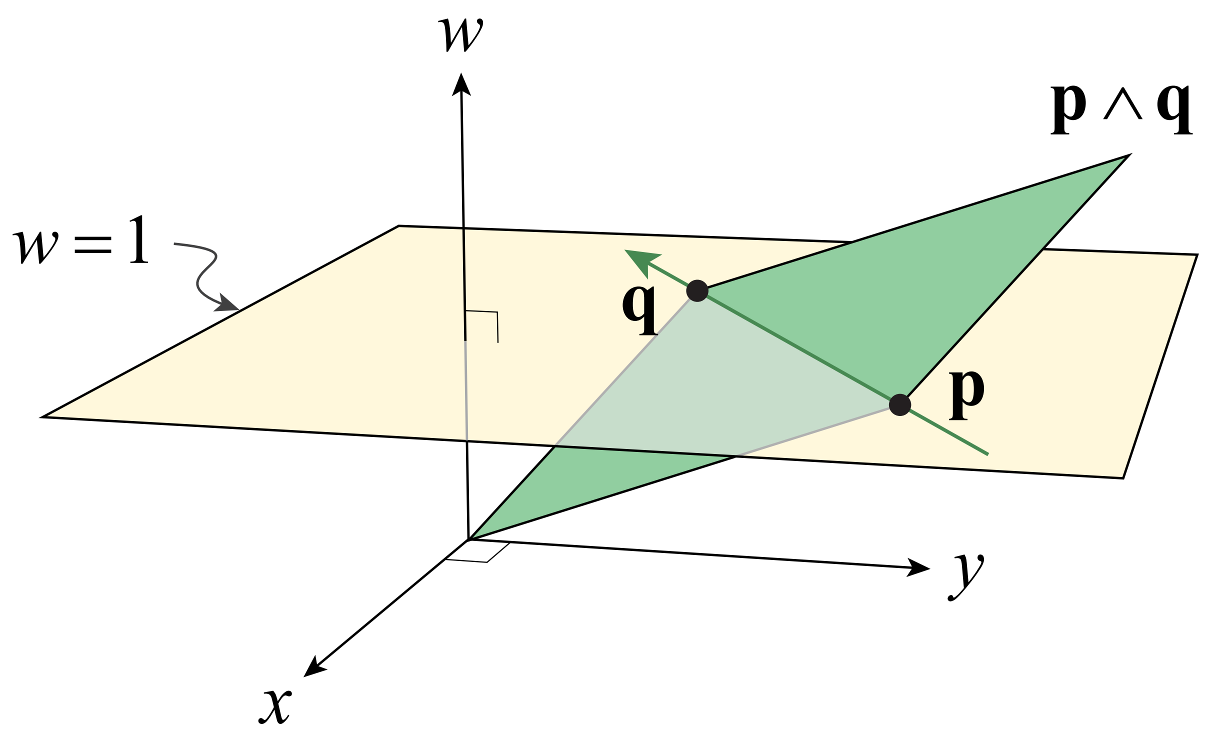
- Wedge product of two points is a bivector

$$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\mathbf{e}_{41} + (q_y p_w - p_y q_w)\mathbf{e}_{42} + (q_z p_w - p_z q_w)\mathbf{e}_{43} + (p_y q_z - p_z q_y)\mathbf{e}_{23} + (p_z q_x - p_x q_z)\mathbf{e}_{31} + (p_x q_y - p_y q_x)\mathbf{e}_{12}$$

$$\mathbf{L} = \underbrace{v_x \mathbf{e}_{41} + v_y \mathbf{e}_{42} + v_z \mathbf{e}_{43}}_{\text{Direction}} + \underbrace{m_x \mathbf{e}_{23} + m_y \mathbf{e}_{31} + m_z \mathbf{e}_{12}}_{\text{Moment}}$$

Homogeneous Line

- Projection of 2D bivector into subspace at $w = 1$ is a 1D line

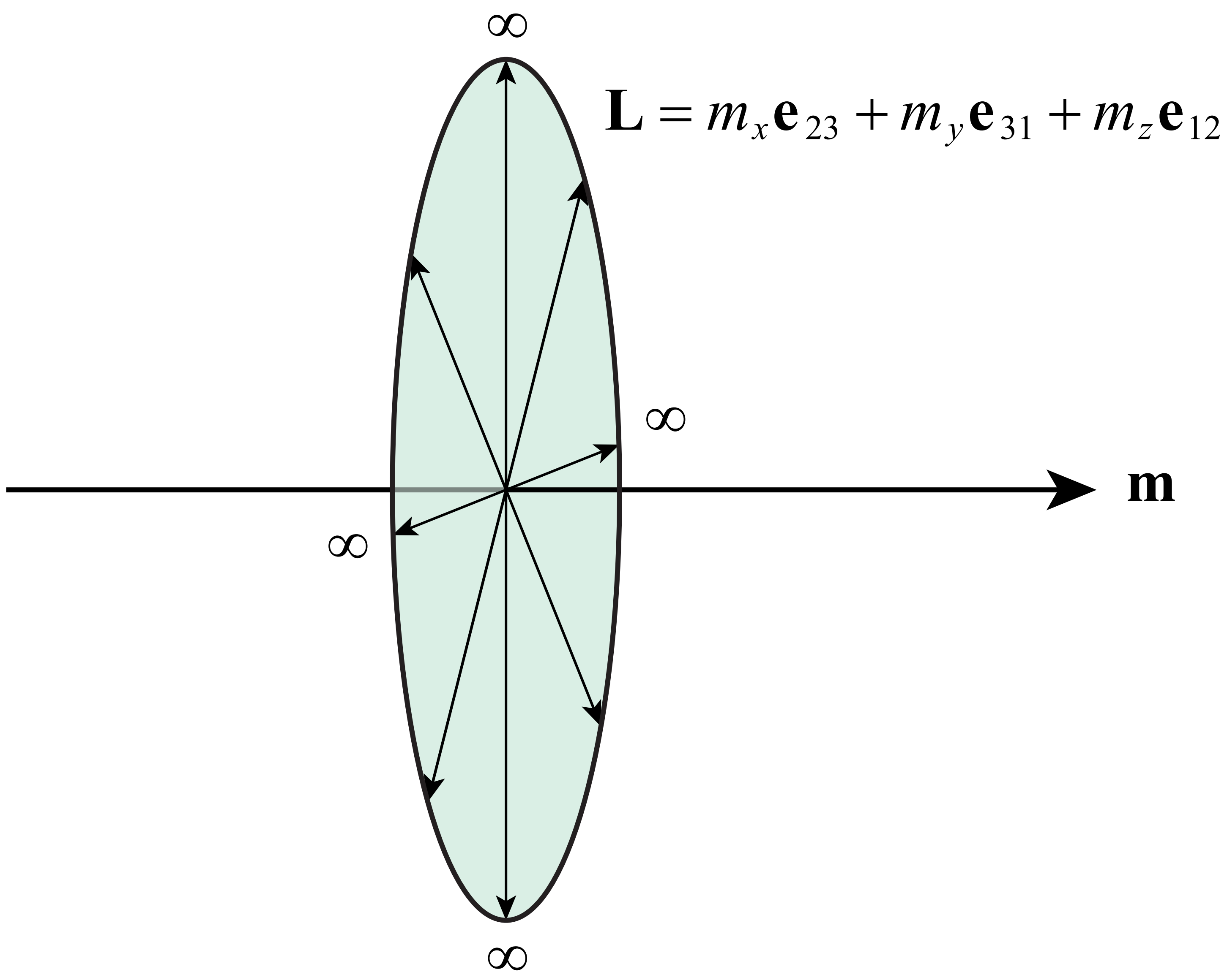


Line at Infinity

- If direction part is zero, then line lies at infinity in directions perpendicular to moment

- Each line at infinity exists in a plane of directions

Line at Infinity



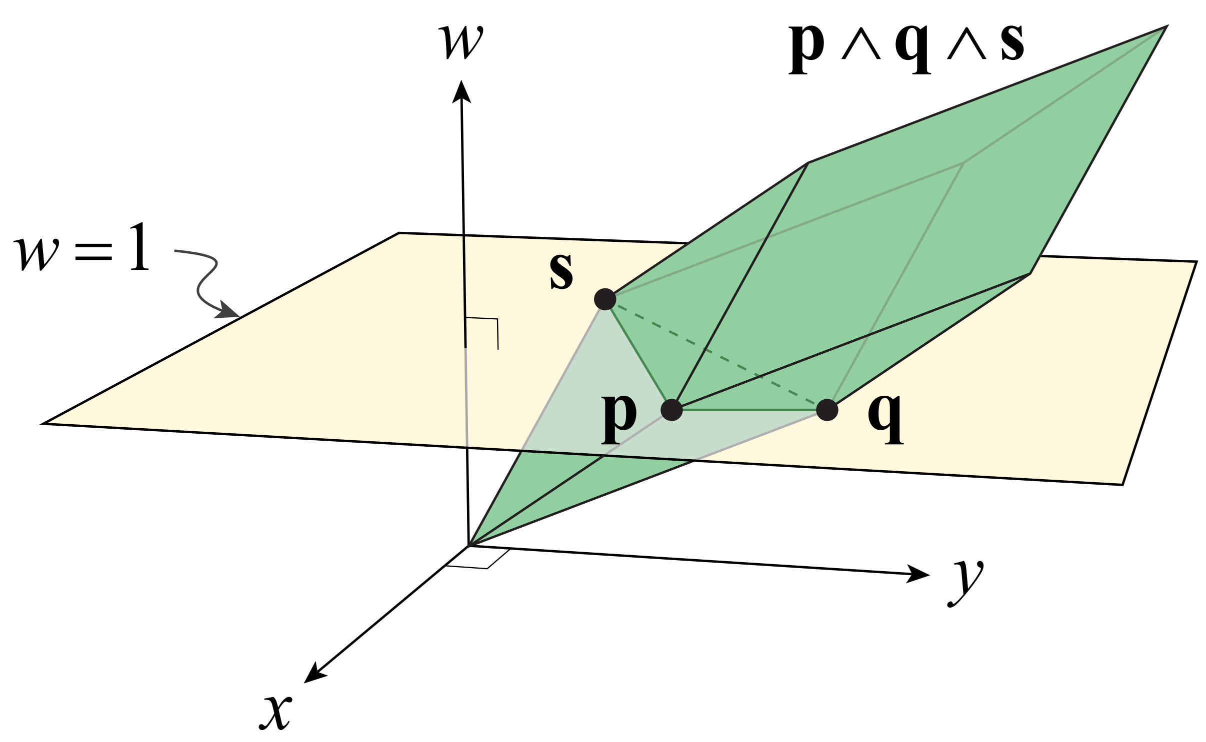
Homogeneous Plane

- Wedge product of three points is a trivector

$$\mathbf{f} = \underbrace{f_x \mathbf{e}_{234} + f_y \mathbf{e}_{314} + f_z \mathbf{e}_{124}}_{\text{Normal}} + f_w \mathbf{e}_{321}$$

Homogeneous Plane

- Projection of 3D trivector into subspace at $w = 1$ is a 2D plane



Plane at Infinity

- There is one plane at infinity
  - Just like there is one point at the origin
  - These are duals of each other

- The plane at infinity exists in all directions

Bulk and Weight

- Components of any object can be separated into two parts
- The “bulk” consists of all components that do not have a factor of $\mathbf{e}_4$
- The “weight” consists of all components that do have a factor of $\mathbf{e}_4$

Bulk and Weight

- The bulk of $\mathbf{a}$ is denoted by $\mathbf{a}_{\bullet}$
- The weight of $\mathbf{a}$ is denoted by $\mathbf{a}_{\circ}$
- Any object is the sum of its bulk and weight:

$$\mathbf{a} = \mathbf{a}_{\bullet} + \mathbf{a}_{\circ}$$

Bulk and Weight

- Weight of point is its *w* coordinate
- Weight of line is its direction
- Weight of plane is its normal

Type	Definition	Bulk	Weight
Point	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$\mathbf{p}_{\bullet} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$	$\mathbf{p}_{\circ} = p_w\mathbf{e}_4$
Line	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\mathbf{L}_{\bullet} = m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\mathbf{L}_{\circ} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43}$
Plane	$\mathbf{f} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124} + f_w\mathbf{e}_{321}$	$\mathbf{f}_{\bullet} = f_w\mathbf{e}_{321}$	$\mathbf{f}_{\circ} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124}$

Bulk and Weight

- The bulk contains an object's position
- The weight contains attitude and orientation

- An object with zero bulk contains the point at the origin

- An object with zero weight is contained by the plane at infinity

Duality

- There is a fundamental symmetry in geometric algebra

- We have assigned dimensionality to objects based on how many basis vectors are *present*

- Objects have another dimensionality based on how many basis vectors are *absent*

Duality

- Every object is really two things at once
  - Full space and empty space
  - Grade and antigrade

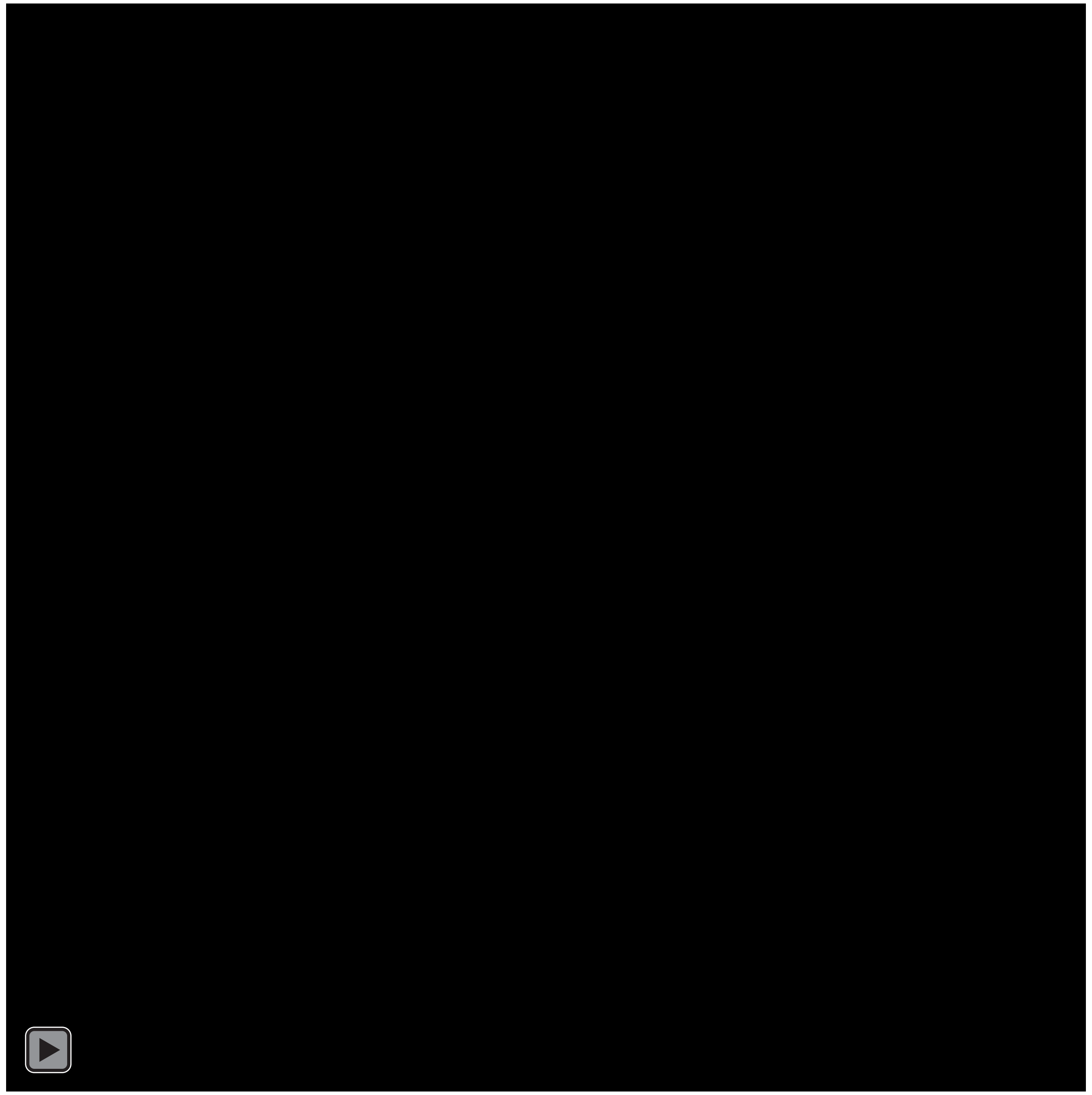
- This is duality, and it's everywhere in GA

Type	Basis Elements	Grade / Antigrade	
Scalar	$\mathbf{1}$	0 / 4	▯▯▯▯
Vectors	$\mathbf{e}_1$	1 / 3	▮▯▯▯
	$\mathbf{e}_2$		▯▮▯▯
	$\mathbf{e}_3$		▯▯▮▯
	$\mathbf{e}_4$		▯▯▯▮
Bivectors	$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$	2 / 2	▯▮▮▯
	$\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$		▮▯▮▯
	$\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$		▮▮▯▯
	$\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$		▯▯▮▮
	$\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$		▯▮▯▮
	$\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$		▮▯▯▮
Trivectors / Antivectors	$\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$	3 / 1	▮▮▮▯
	$\mathbf{e}_{124} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$		▮▮▯▮
	$\mathbf{e}_{314} = \mathbf{e}_3 \wedge \mathbf{e}_1 \wedge \mathbf{e}_4$		▮▯▮▮
	$\mathbf{e}_{234} = \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$		▯▮▮▮
Antiscalar	$\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$	4 / 0	▮▮▮▮

Duality

- A point has one full dimension
- It also has three empty dimensions
- From different perspectives, it simultaneously looks like a point and a plane

Duality



Dualization

- We can map basis elements so that full and empty dimensions are exchanged
- If we think of the dimensions used by a basis element as a 4-bit code, then dualization inverts the bits
- There are many choices for dualization functions, and they just differ in sign in a grade-dependent manner

Complements

- One choice for dualization is the "right complement"

- The right complement of $\mathbf{a}$ is the object $\overline{\mathbf{a}}$ such that

$$\mathbf{a} \wedge \overline{\mathbf{a}} = \mathbb{1}$$

- This is also called the Hodge dual

Complements

- In 4D, right complement is not an involution

- The inverse is the “left complement” $\underline{\mathbf{a}}$

$$\underline{\mathbf{a}} \wedge \mathbf{a} = \mathbb{1}$$

- Right and left complements differ only in sign

Complements

- Here, the basis elements are ordered so that taking the complement just reverses the list and adjusts the sign

Basis element $\mathbf{a}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{124}$	$\mathbf{e}_{314}$	$\mathbf{e}_{234}$	$\mathbb{1}$
Right complement $\overline{\mathbf{a}}$	$\mathbb{1}$	$\mathbf{e}_{234}$	$\mathbf{e}_{314}$	$\mathbf{e}_{124}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$-\mathbf{e}_1$	$\mathbf{1}$
Left complement $\underline{\mathbf{a}}$	$\mathbb{1}$	$-\mathbf{e}_{234}$	$-\mathbf{e}_{314}$	$-\mathbf{e}_{124}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{e}_4$	$\mathbf{e}_3$	$\mathbf{e}_2$	$\mathbf{e}_1$	$\mathbf{1}$
Double complement $\overline{\overline{\mathbf{a}}}$ or $\underline{\underline{\mathbf{a}}}$	$\mathbf{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{124}$	$-\mathbf{e}_{314}$	$-\mathbf{e}_{234}$	$\mathbb{1}$

Attitude Extraction

- Weight contains information about attitude
- The weight complement is useful for extracting this information to be used another way
- Very useful for projections

Type	Definition	Weight Complement	Interpretation
Point	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$\underline{\mathbf{p}_{\bigcirc}} = -p_w\mathbf{e}_{321}$	Plane at infinity.
Line	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\underline{\mathbf{L}_{\bigcirc}} = -v_x\mathbf{e}_{23} - v_y\mathbf{e}_{31} - v_z\mathbf{e}_{12}$	Line at infinity perpendicular to line **L**.
Plane	$\mathbf{f} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124} + f_w\mathbf{e}_{321}$	$\underline{\mathbf{f}_{\bigcirc}} = f_x\mathbf{e}_1 + f_y\mathbf{e}_2 + f_z\mathbf{e}_3$	Normal vector of the plane **f**.

Antiwedge Product

- Also known as exterior antiproduct
  - Grassmann called it regressive combinatorial product

- Written with downward wedge:

$$\mathbf{a} \vee \mathbf{b}$$

- Read as "$\mathbf{a}$ antiwedge $\mathbf{b}$"

Antiwedge Product

- Wedge product combines full dimensions
  - Add grades of operands
- Antiwedge product combines empty dimensions
  - Add antigrades of operands

Antiwedge Product

- Dual to wedge product

$$\mathbf{c} = \mathbf{a} \wedge \mathbf{b}$$

$$\overline{\mathbf{c}} = \overline{\mathbf{a}} \vee \overline{\mathbf{b}}$$

- Operates on antivectors in same way that wedge product operates on vectors

De Morgan’s Laws

- All operations in GA have duals that together satisfy De Morgan’s Laws
- For wedge and antiwedge:

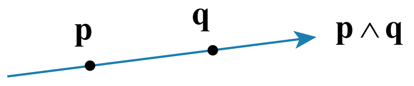
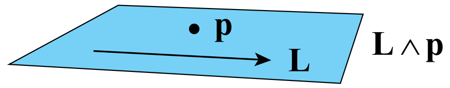
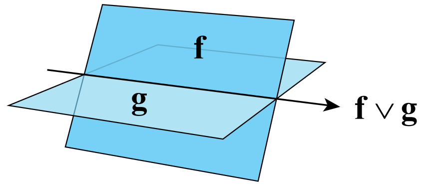
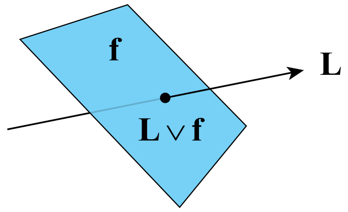
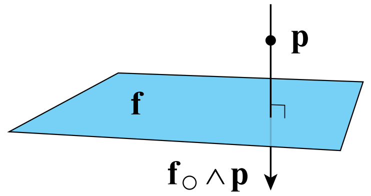
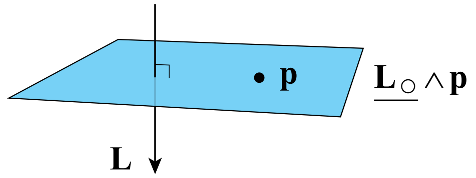
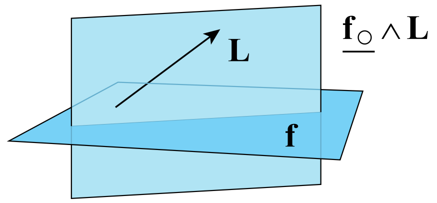
$$\mathbf{a} \vee \mathbf{b} = \underline{\overline{\mathbf{a}} \wedge \overline{\mathbf{b}}} = \overline{\underline{\mathbf{a}} \wedge \underline{\mathbf{b}}}$$

- This can be taken as definition of antiwedge
  - Depends on specific choice of dualization function
  - Only affects orientation of some results

Join and Meet

- Wedge product combines full dimensions
  - Join operation
  - Analogous to union

- Antiwedge product combines empty dimensions
  - Meet operation
  - Analogous to intersection

Formula	Description	Illustration
$\mathbf{p}\wedge\mathbf{q}=(q_x p_w - p_x q_w)\,\mathbf{e}_{41}+(q_y p_w - p_y q_w)\,\mathbf{e}_{42}+(q_z p_w - p_z q_w)\,\mathbf{e}_{43}+(p_y q_z - p_z q_y)\,\mathbf{e}_{23}+(p_z q_x - p_x q_z)\,\mathbf{e}_{31}+(p_x q_y - p_y q_x)\,\mathbf{e}_{12}$	Line containing points **p** and **q**. Zero if **p** and **q** are coincident.	
$\mathbf{L}\wedge\mathbf{p}=(v_y p_z - v_z p_y + m_x p_w)\,\mathbf{e}_{234}+(v_z p_x - v_x p_z + m_y p_w)\,\mathbf{e}_{314}+(v_x p_y - v_y p_x + m_z p_w)\,\mathbf{e}_{124}-(m_x p_x + m_y p_y + m_z p_z)\,\mathbf{e}_{321}$	Plane containing line **L** and point **p**. Normal is zero if **p** lies in **L**.	
$\mathbf{f}\vee\mathbf{g}=(f_z g_y - f_y g_z)\,\mathbf{e}_{41}+(f_x g_z - f_z g_x)\,\mathbf{e}_{42}+(f_y g_x - f_x g_y)\,\mathbf{e}_{43}+(f_x g_w - g_x f_w)\,\mathbf{e}_{23}+(f_y g_w - g_y f_w)\,\mathbf{e}_{31}+(f_z g_w - g_z f_w)\,\mathbf{e}_{12}$	Line where planes **f** and **g** intersect. Direction is zero if **f** and **g** are parallel.	
$\mathbf{L}\vee\mathbf{f}=(m_y f_z - m_z f_y + v_x f_w)\,\mathbf{e}_{1}+(m_z f_x - m_x f_z + v_y f_w)\,\mathbf{e}_{2}+(m_x f_y - m_y f_x + v_z f_w)\,\mathbf{e}_{3}-(v_x f_x + v_y f_y + v_z f_z)\,\mathbf{e}_{4}$	Point where line **L** intersects plane **f**. Weight is zero if **L** and **f** are parallel.	
$\underline{\mathbf{f}_{\circ}}\wedge\mathbf{p}=-f_x p_w\mathbf{e}_{41}-f_y p_w\mathbf{e}_{42}-f_z p_w\mathbf{e}_{43}+(f_y p_z - f_z p_y)\,\mathbf{e}_{23}+(f_z p_x - f_x p_z)\,\mathbf{e}_{31}+(f_x p_y - f_y p_x)\,\mathbf{e}_{12}$	Line perpendicular to plane **f** and passing through point **p**.	
$\underline{\mathbf{L}_{\circ}}\wedge\mathbf{p}=-v_x p_w\mathbf{e}_{234}-v_y p_w\mathbf{e}_{314}-v_z p_w\mathbf{e}_{124}+(v_x p_x + v_y p_y + v_z p_z)\,\mathbf{e}_{321}$	Plane perpendicular to line **L** and containing point **p**.	
$\underline{\mathbf{f}_{\circ}}\wedge\mathbf{L}=(v_y f_z - v_z f_y)\,\mathbf{e}_{234}+(v_z f_x - v_x f_z)\,\mathbf{e}_{314}+(v_x f_y - v_y f_x)\,\mathbf{e}_{124}-(m_x f_x + m_y f_y + m_z f_z)\,\mathbf{e}_{321}$	Plane perpendicular to plane **f** and containing line **L**. Normal is zero if **L** is perpendicular to **f**.	

Plane/Point Volume

- Wedge product of point $\mathbf{p}$ and plane $\mathbf{f}$ is

$$\mathbf{p} \wedge \mathbf{f} = (p_x f_x + p_y f_y + p_z f_z + p_w f_w)\,\mathbb{1}$$

- Same as conventional dot product

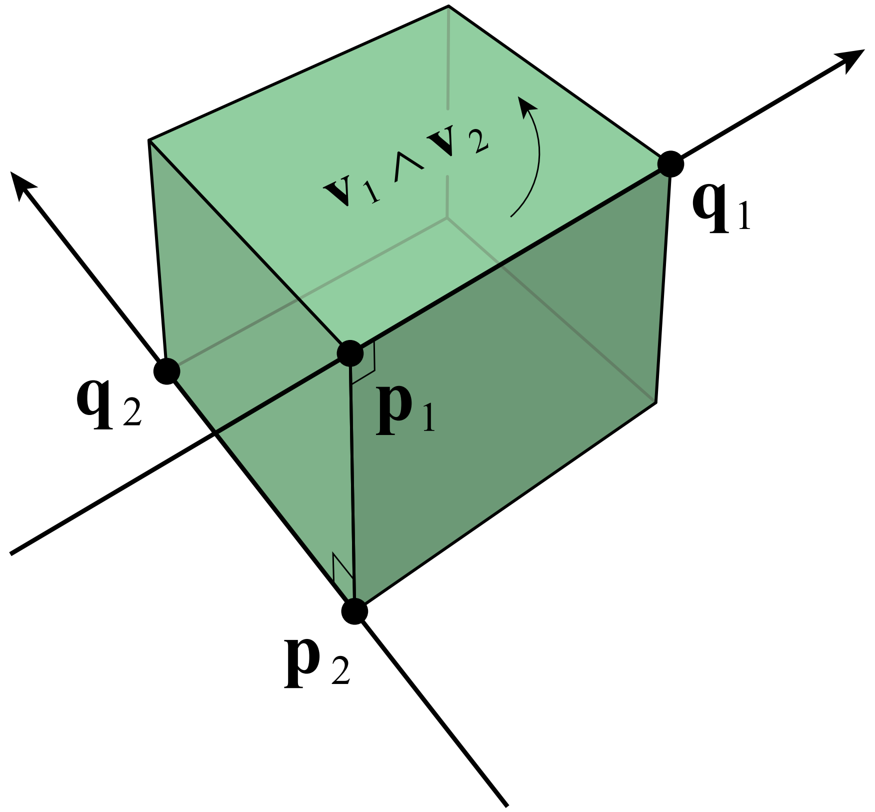
- Gives signed distance between point and plane, scaled by weights of point and plane

Line/Line Volume

- Wedge product of two lines $\mathbf{L}_1$ and $\mathbf{L}_2$ is

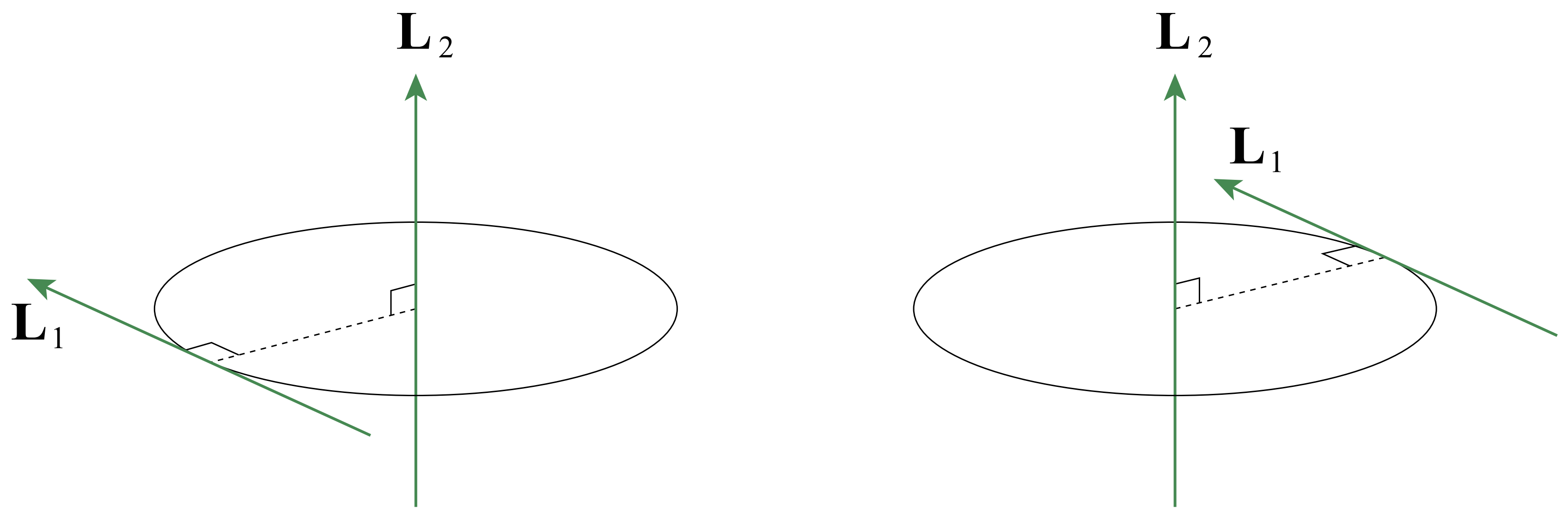
$$\mathbf{L}_1 \wedge \mathbf{L}_2 = -(\mathbf{v}_1 \cdot \mathbf{m}_2 + \mathbf{v}_2 \cdot \mathbf{m}_1)\mathbb{1}$$

- Gives signed distance between lines, scaled by magnitude of $\mathbf{v}_1 \wedge \mathbf{v}_2$



Line Crossing

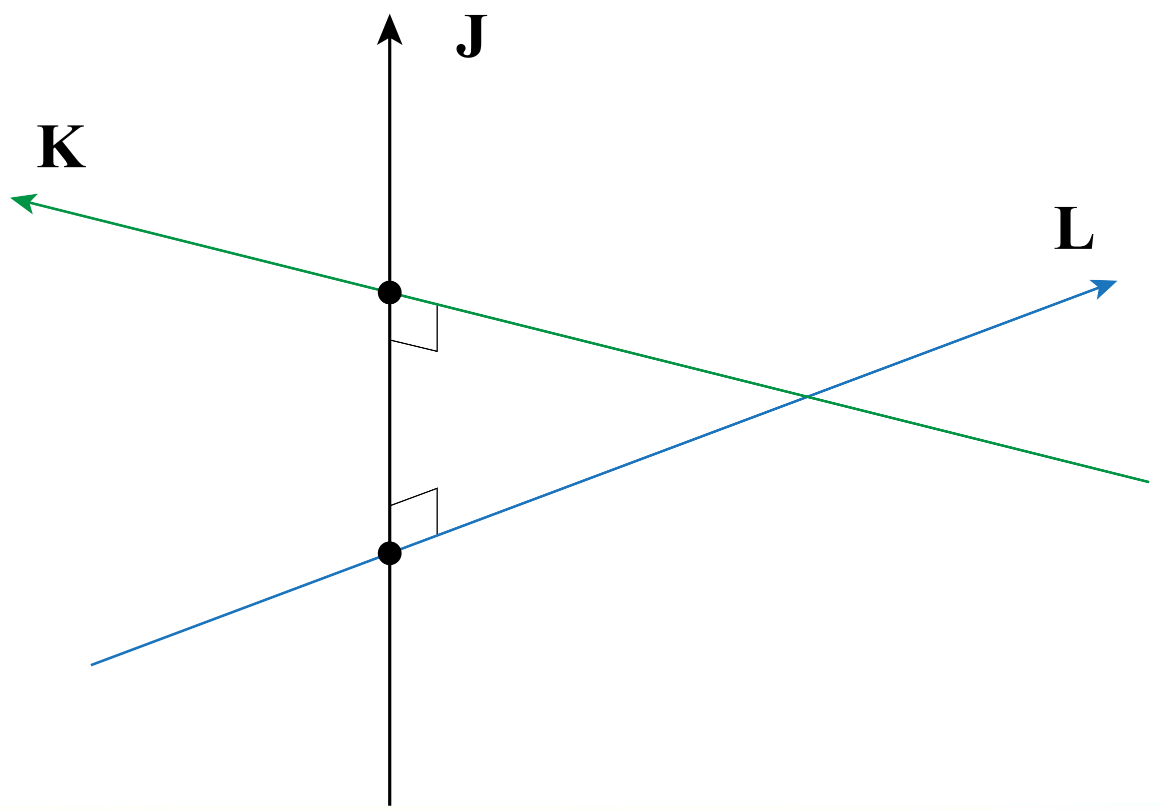
- Antiwedge product gives same value as scalar
- Used to detect which way lines cross each other



$\mathbf{L}_1 \vee \mathbf{L}_2 > 0$ $\mathbf{L}_1 \vee \mathbf{L}_2 < 0$

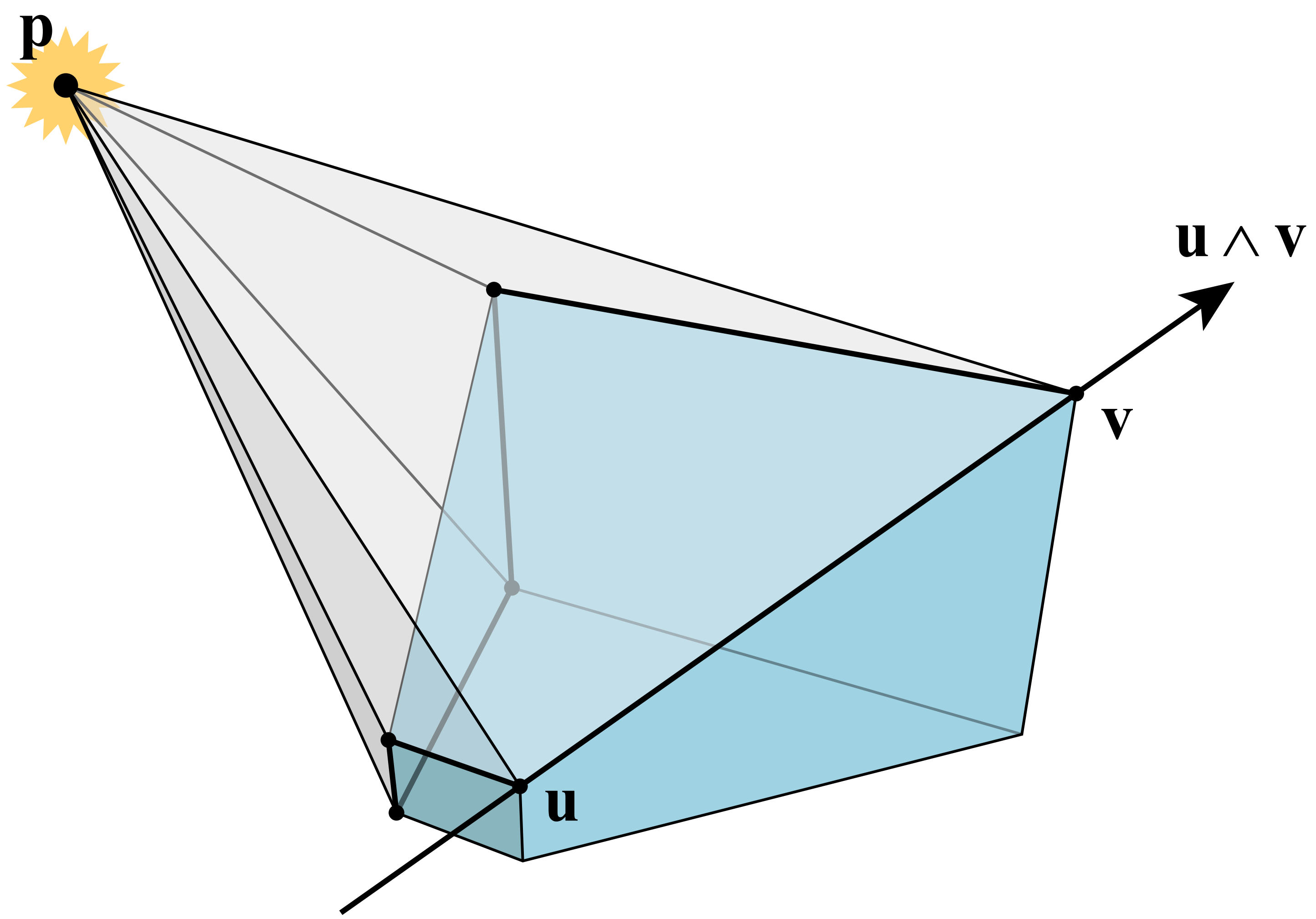
Line Between Two Lines

- Line $\mathbf{J}$ perpendicular to lines $\mathbf{K}$ and $\mathbf{L}$
  - Can't be produced by wedge/antiwedge product
  - It does appear in the geometric product



Application: Shadow Regions

- Need convex region where shadow castors must be to affect scene
- Precompute lines for frustum edges
- Find silhouette w.r.t. light
- Take wedge products with light position

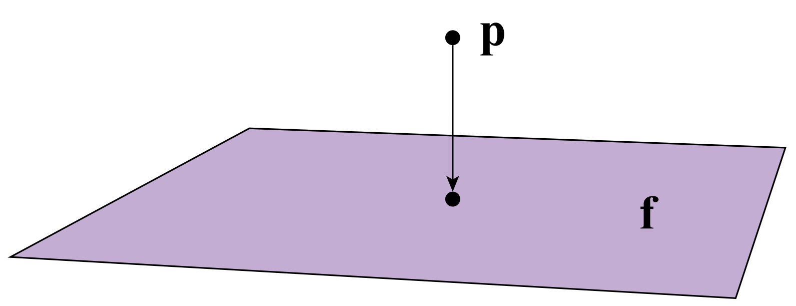
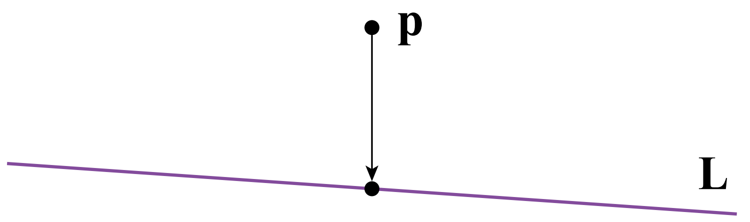
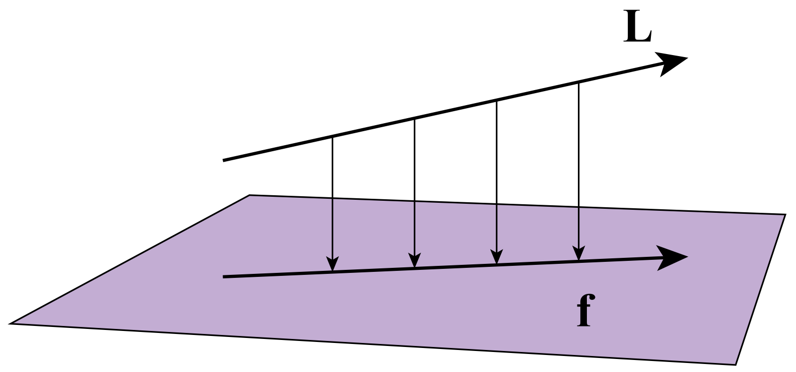


Projections

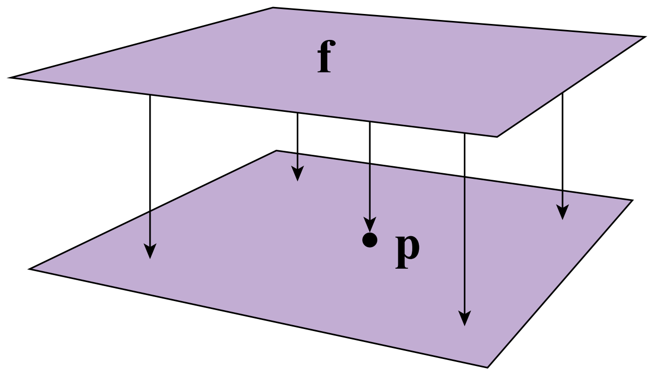
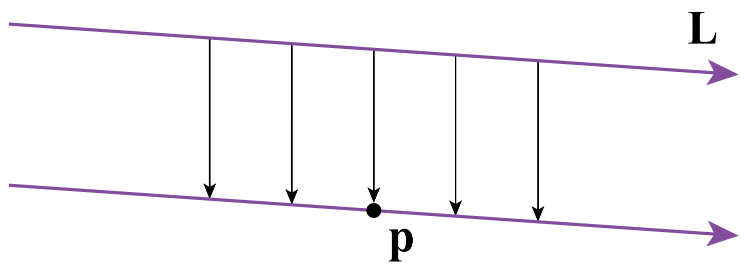
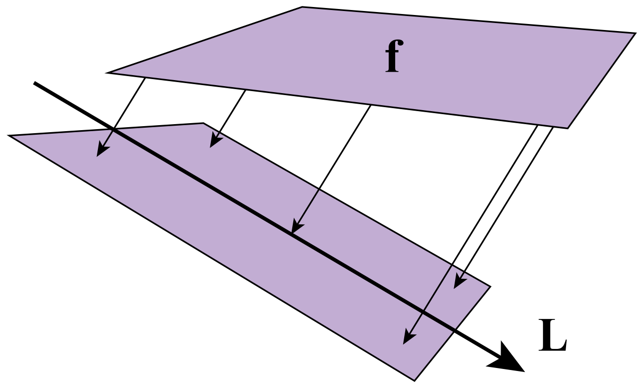
- Wedge and antiwedge products in specific combinations perform projections

- These are derived from "interior products"
  - All projections have a uniform formula
  - Interior antiproducts perform "antiprojections"

Projections

Projection	Illustration
Projection of point **p** onto plane **f**. $(\underline{\mathbf{f}_\circ} \wedge \mathbf{p}) \vee \mathbf{f} = (f_x^2 + f_y^2 + f_z^2)\mathbf{p} - (f_x p_x + f_y p_y + f_z p_z + f_w p_w)(f_x \mathbf{e}_1 + f_y \mathbf{e}_2 + f_z \mathbf{e}_3)$	
Projection of point **p** onto line **L**. $(\underline{\mathbf{L}_\circ} \wedge \mathbf{p}) \vee \mathbf{L} = (v_x p_x + v_y p_y + v_z p_z)\mathbf{v} + (v_x^2 + v_y^2 + v_z^2) p_w \mathbf{e}_4$ $+ (v_y m_z - v_z m_y) p_w \mathbf{e}_1 + (v_z m_x - v_x m_z) p_w \mathbf{e}_2 + (v_x m_y - v_y m_x) p_w \mathbf{e}_3$	
Projection of line **L** onto plane **f**. $(\underline{\mathbf{f}_\circ} \wedge \mathbf{L}) \vee \mathbf{f} = (f_x^2 + f_y^2 + f_z^2)(v_x \mathbf{e}_{41} + v_y \mathbf{e}_{42} + v_z \mathbf{e}_{43})$ $- (f_x v_x + f_y v_y + f_z v_z)(f_x \mathbf{e}_{41} + f_y \mathbf{e}_{42} + f_z \mathbf{e}_{43})$ $+ (f_x m_x + f_y m_y + f_z m_z)(f_x \mathbf{e}_{23} + f_y \mathbf{e}_{31} + f_z \mathbf{e}_{12})$ $- (f_y v_z - f_z v_y) f_w \mathbf{e}_{23} - (f_z v_x - f_x v_z) f_w \mathbf{e}_{31} - (f_x v_y - f_y v_x) f_w \mathbf{e}_{12}$	

Antiprojections

Antiprojection	Illustration
Antiprojection of plane **f** onto point **p**. $\left(\underline{\mathbf{p}_{\circ}} \vee \mathbf{f}\right) \wedge \mathbf{p} = f_x p_w^2 \mathbf{e}_{234} + f_y p_w^2 \mathbf{e}_{314} + f_z p_w^2 \mathbf{e}_{124} - (f_x p_x + f_y p_y + f_z p_z)\, p_w \mathbf{e}_{321}$	
Antiprojection of line **L** onto point **p**. $\left(\underline{\mathbf{p}_{\circ}} \vee \mathbf{L}\right) \wedge \mathbf{p} = v_x p_w^2 \mathbf{e}_{41} + v_y p_w^2 \mathbf{e}_{42} + v_z p_w^2 \mathbf{e}_{43}$ $+ (p_y v_z - p_z v_y)\, p_w \mathbf{e}_{23} + (p_z v_x - p_x v_z)\, p_w \mathbf{e}_{31} + (p_x x_y - p_y v_x)\, p_w \mathbf{e}_{12}$	
Antiprojection of plane **f** onto line **L**. $\left(\underline{\mathbf{L}_{\circ}} \vee \mathbf{f}\right) \wedge \mathbf{L} = (v_x^2 + v_y^2 + v_z^2)(f_x \mathbf{e}_{234} + f_y \mathbf{e}_{314} + f_z \mathbf{e}_{124})$ $- (f_x v_x + f_y v_y + f_z v_z)(v_x \mathbf{e}_{234} + v_y \mathbf{e}_{314} + v_z \mathbf{e}_{124})$ $+ (f_x m_y v_z - f_x m_z v_y + f_y m_z v_x - f_y m_x v_z + f_z m_x v_y - f_z m_y v_x)\, \mathbf{e}_{321}$	

Special Projections

- Point at origin and plane at infinity produce special values

Projection	Description
$(\underline{\mathbf{f}_{\circ}} \wedge \mathbf{e}_4) \vee \mathbf{f} = -f_x f_w \mathbf{e}_1 - f_y f_w \mathbf{e}_2 - f_z f_w \mathbf{e}_3 + (f_x^2 + f_y^2 + f_z^2)\,\mathbf{e}_4$	Point closest to the origin on the plane **f**.
$(\underline{\mathbf{L}_{\circ}} \wedge \mathbf{e}_4) \vee \mathbf{L} = (v_y m_z - v_z m_y)\,\mathbf{e}_1 + (v_z m_x - v_x m_z)\,\mathbf{e}_2 + (v_x m_y - v_y m_x)\,\mathbf{e}_3 + (v_x^2 + v_y^2 + v_z^2)\,\mathbf{e}_4$	Point closest to the origin on the line **L**.

Antiprojection	Description
$(\underline{\mathbf{p}_{\bullet}} \vee \mathbf{e}_{321}) \wedge \mathbf{p} = -p_x p_w \mathbf{e}_{234} - p_y p_w \mathbf{e}_{314} - p_z p_w \mathbf{e}_{124} + (p_x^2 + p_y^2 + p_z^2)\,\mathbf{e}_{321}$	Plane farthest from the origin containing the point **p**.
$(\underline{\mathbf{L}_{\bullet}} \vee \mathbf{e}_{321}) \wedge \mathbf{L} = (m_y v_z - m_z v_y)\,\mathbf{e}_{234} + (m_z v_x - m_x v_z)\,\mathbf{e}_{314} + (m_x v_y - m_y v_x)\,\mathbf{e}_{124} + (m_x^2 + m_y^2 + m_z^2)\,\mathbf{e}_{321}$	Plane farthest from the origin containing the line **L**.

Geometric Product

- Adds more information to wedge product
- Incorporates a metric
  - Allows us to make measurements
  - Provides the mechanism for Euclidean isometries

- Like all operations in GA, the geometric product has a dual operation, or *antiproduct*

Geometric Product

- Conventional treatments of GA ignore the antiproduct
- Geometric product has been expressed by plain old juxtaposition: $\mathbf{c} = \mathbf{ab}$
- With two products, we need an infix symbol to distinguish between them

Geometric Product

- The geometric product incorporates the wedge product and adds information to it
- So we write the geometric product as

$$\mathbf{a} \mathbin{\dot{\wedge}} \mathbf{b}$$

- We read this as "**a** wedge-dot **b**"

Geometric Antiproduct

- The geometric antiproduct incorporates the antiwedge product and adds information to it
- So we write the geometric antiproduct as

$$\mathbf{a} \mathbin{\dot\vee} \mathbf{b}$$

- We read this as "**a** antiwedge-dot **b**"

Metric

- The 4D projective geometric algebra is denoted by $\mathcal{G}_{3,0,1}$
- The subscripts mean that:
  - 3 basis vectors square to $+1$
  - 0 basis vectors square to -1
  - 1 basis vector squares to 0
- The fourth dimension has no physical measure

Metric

- Metrics apply symmetrically to geometric product and antiproduct

$$\mathbf{e}_1 \mathbin{\dot\wedge} \mathbf{e}_1 = 1 \qquad \overline{\mathbf{e}}_1 \mathbin{\dot\vee} \overline{\mathbf{e}}_1 = \mathbb{1}$$

$$\mathbf{e}_2 \mathbin{\dot\wedge} \mathbf{e}_2 = 1 \qquad \overline{\mathbf{e}}_2 \mathbin{\dot\vee} \overline{\mathbf{e}}_2 = \mathbb{1}$$

$$\mathbf{e}_3 \mathbin{\dot\wedge} \mathbf{e}_3 = 1 \qquad \overline{\mathbf{e}}_3 \mathbin{\dot\vee} \overline{\mathbf{e}}_3 = \mathbb{1}$$

$$\mathbf{e}_4 \mathbin{\dot\wedge} \mathbf{e}_4 = 0 \qquad \overline{\mathbf{e}}_4 \mathbin{\dot\vee} \overline{\mathbf{e}}_4 = 0$$

Geometric Product

Geometric Product $\mathbf{a} ⟑ \mathbf{b}$

☐ $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} ⟑ \mathbf{b}$ ☐ (shaded) $\mathbf{a} \wedge \mathbf{b} = 0$

a \ b	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{124}$	$\mathbf{e}_{314}$	$\mathbf{e}_{234}$	$\mathbb{1}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{4}$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{124}$	$\mathbf{e}_{314}$	$\mathbf{e}_{234}$	$\mathbb{1}$
$\mathbf{e}_{1}$	$\mathbf{e}_{1}$	$\mathbf{1}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbf{e}_{314}$	$-\mathbf{e}_{124}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{43}$	$\mathbb{1}$	$\mathbf{e}_{234}$
$\mathbf{e}_{2}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{12}$	$\mathbf{1}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$-\mathbf{e}_{234}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{124}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{41}$	$\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{314}$
$\mathbf{e}_{3}$	$\mathbf{e}_{3}$	$\mathbf{e}_{31}$	$-\mathbf{e}_{23}$	$\mathbf{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{234}$	$-\mathbf{e}_{314}$	$-\mathbf{e}_{12}$	$\mathbb{1}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{124}$
$\mathbf{e}_{4}$	$\mathbf{e}_{4}$	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	$\mathbf{e}_{234}$	$\mathbf{e}_{314}$	$\mathbf{e}_{124}$	0	0	0	$\mathbb{1}$	0	0	0	0
$\mathbf{e}_{23}$	$\mathbf{e}_{23}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{3}$	$\mathbf{e}_{2}$	$\mathbf{e}_{234}$	$-\mathbf{1}$	$-\mathbf{e}_{12}$	$\mathbf{e}_{31}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{43}$	$-\mathbb{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{314}$	$-\mathbf{e}_{124}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{41}$
$\mathbf{e}_{31}$	$\mathbf{e}_{31}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{1}$	$\mathbf{e}_{314}$	$\mathbf{e}_{12}$	$-\mathbf{1}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{41}$	$-\mathbb{1}$	$\mathbf{e}_{43}$	$\mathbf{e}_{2}$	$-\mathbf{e}_{234}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{124}$	$\mathbf{e}_{42}$
$\mathbf{e}_{12}$	$\mathbf{e}_{12}$	$-\mathbf{e}_{2}$	$\mathbf{e}_{1}$	$-\mathbf{e}_{321}$	$\mathbf{e}_{124}$	$-\mathbf{e}_{31}$	$\mathbf{e}_{23}$	$-\mathbf{1}$	$-\mathbb{1}$	$\mathbf{e}_{41}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{3}$	$-\mathbf{e}_{4}$	$\mathbf{e}_{234}$	$-\mathbf{e}_{314}$	$\mathbf{e}_{43}$
$\mathbf{e}_{43}$	$\mathbf{e}_{43}$	$\mathbf{e}_{314}$	$-\mathbf{e}_{234}$	$\mathbf{e}_{4}$	0	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	0	0	$-\mathbf{e}_{124}$	0	0	0	0
$\mathbf{e}_{42}$	$\mathbf{e}_{42}$	$-\mathbf{e}_{124}$	$\mathbf{e}_{4}$	$\mathbf{e}_{234}$	0	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	0	0	$-\mathbf{e}_{314}$	0	0	0	0
$\mathbf{e}_{41}$	$\mathbf{e}_{41}$	$\mathbf{e}_{4}$	$\mathbf{e}_{124}$	$-\mathbf{e}_{314}$	0	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	0	0	$-\mathbf{e}_{234}$	0	0	0	0
$\mathbf{e}_{321}$	$\mathbf{e}_{321}$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbb{1}$	$\mathbf{e}_{1}$	$\mathbf{e}_{2}$	$\mathbf{e}_{3}$	$\mathbf{e}_{124}$	$\mathbf{e}_{314}$	$\mathbf{e}_{234}$	$-\mathbf{1}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{4}$
$\mathbf{e}_{124}$	$\mathbf{e}_{124}$	$-\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$-\mathbb{1}$	0	$-\mathbf{e}_{314}$	$\mathbf{e}_{234}$	$-\mathbf{e}_{4}$	0	0	0	$\mathbf{e}_{43}$	0	0	0	0
$\mathbf{e}_{314}$	$\mathbf{e}_{314}$	$\mathbf{e}_{43}$	$-\mathbb{1}$	$-\mathbf{e}_{41}$	0	$\mathbf{e}_{124}$	$-\mathbf{e}_{4}$	$-\mathbf{e}_{234}$	0	0	0	$\mathbf{e}_{42}$	0	0	0	0
$\mathbf{e}_{234}$	$\mathbf{e}_{234}$	$-\mathbb{1}$	$-\mathbf{e}_{43}$	$\mathbf{e}_{42}$	0	$-\mathbf{e}_{4}$	$-\mathbf{e}_{124}$	$\mathbf{e}_{314}$	0	0	0	$\mathbf{e}_{41}$	0	0	0	0
$\mathbb{1}$	$\mathbb{1}$	$-\mathbf{e}_{234}$	$-\mathbf{e}_{314}$	$-\mathbf{e}_{124}$	0	$\mathbf{e}_{41}$	$\mathbf{e}_{42}$	$\mathbf{e}_{43}$	0	0	0	$-\mathbf{e}_{4}$	0	0	0	0

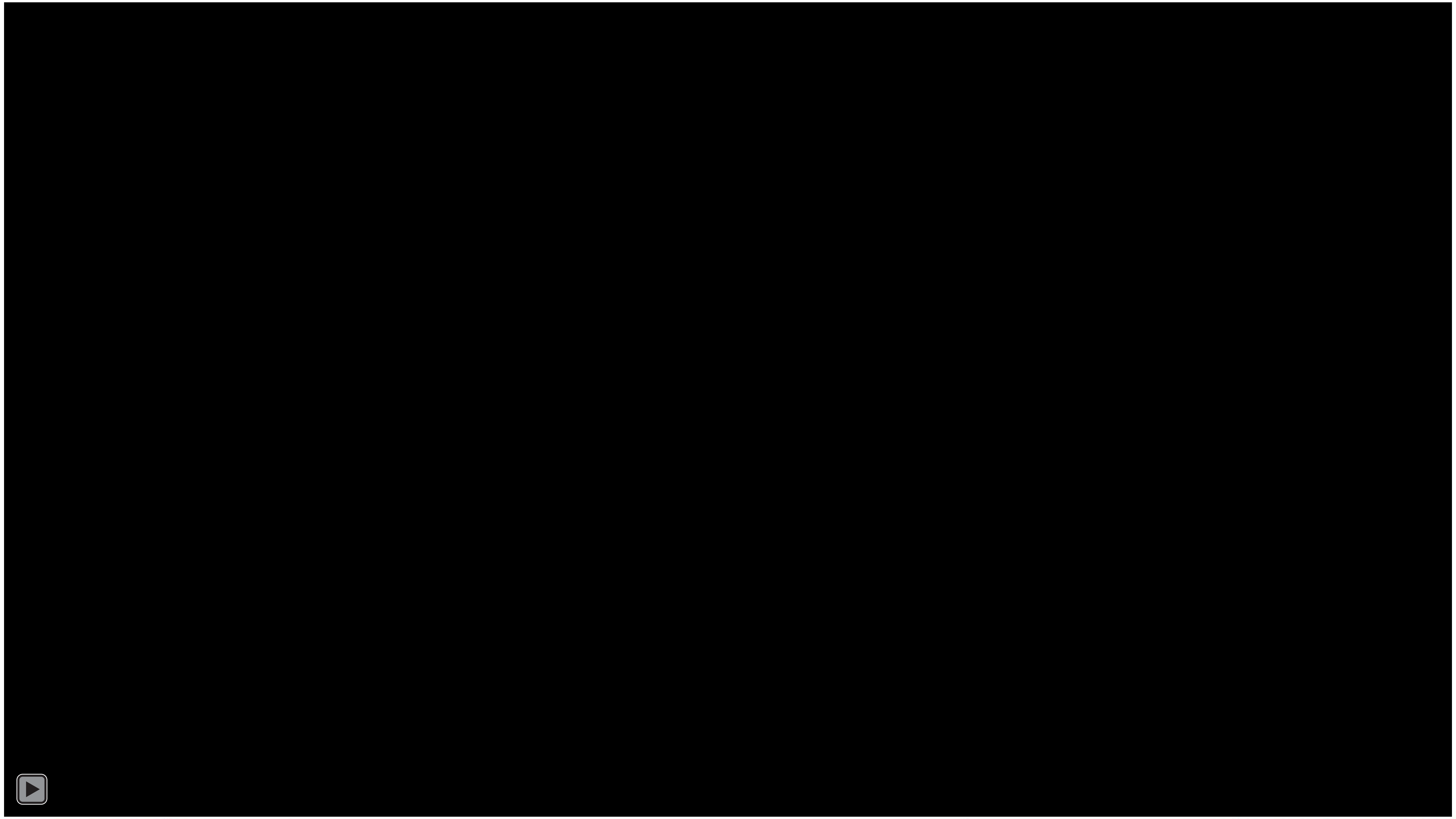
Geometric Antiproduct

Geometric Antiproduct $\mathbf{a} \mathbin{\dot\vee} \mathbf{b}$

☐ $\mathbf{a} \vee \mathbf{b} = \mathbf{a} \mathbin{\dot\vee} \mathbf{b}$ ☐ $\mathbf{a} \vee \mathbf{b} = 0$

a \ b	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{23}	e_{31}	e_{12}	e_{43}	e_{42}	e_{41}	e_{321}	e_{124}	e_{314}	e_{234}	$\mathbb{1}$
1	0	0	0	0	e_{321}	0	0	0	e_{12}	e_{31}	e_{23}	0	e_{3}	e_{2}	e_{1}	1
e_{1}	0	0	0	0	$-e_{23}$	0	0	0	$-e_{2}$	e_{3}	$-e_{321}$	0	e_{31}	$-e_{12}$	1	e_{1}
e_{2}	0	0	0	0	$-e_{31}$	0	0	0	e_{1}	$-e_{321}$	$-e_{3}$	0	$-e_{23}$	1	e_{12}	e_{2}
e_{3}	0	0	0	0	$-e_{12}$	0	0	0	$-e_{321}$	$-e_{1}$	e_{2}	0	1	e_{23}	$-e_{31}$	e_{3}
e_{4}	$-e_{321}$	e_{23}	e_{31}	e_{12}	$-\mathbb{1}$	$-e_{1}$	$-e_{2}$	$-e_{3}$	e_{124}	e_{314}	e_{234}	1	$-e_{43}$	$-e_{42}$	$-e_{41}$	e_{4}
e_{23}	0	0	0	0	e_{1}	0	0	0	$-e_{31}$	e_{12}	-1	0	$-e_{2}$	e_{3}	$-e_{321}$	e_{23}
e_{31}	0	0	0	0	e_{2}	0	0	0	e_{23}	-1	$-e_{12}$	0	e_{1}	$-e_{321}$	$-e_{3}$	e_{31}
e_{12}	0	0	0	0	e_{3}	0	0	0	-1	$-e_{23}$	e_{31}	0	$-e_{321}$	$-e_{1}$	e_{2}	e_{12}
e_{43}	e_{12}	e_{2}	$-e_{1}$	$-e_{321}$	e_{124}	e_{31}	$-e_{23}$	-1	$-\mathbb{1}$	$-e_{41}$	e_{42}	e_{3}	$-e_{4}$	$-e_{234}$	e_{314}	e_{43}
e_{42}	e_{31}	$-e_{3}$	$-e_{321}$	e_{1}	e_{314}	$-e_{12}$	-1	e_{23}	e_{41}	$-\mathbb{1}$	$-e_{43}$	e_{2}	e_{234}	$-e_{4}$	$-e_{124}$	e_{42}
e_{41}	e_{23}	$-e_{321}$	e_{3}	$-e_{2}$	e_{234}	-1	e_{12}	$-e_{31}$	$-e_{42}$	e_{43}	$-\mathbb{1}$	e_{1}	$-e_{314}$	e_{124}	$-e_{4}$	e_{41}
e_{321}	0	0	0	0	-1	0	0	0	e_{3}	e_{2}	e_{1}	0	$-e_{12}$	$-e_{31}$	$-e_{23}$	e_{321}
e_{124}	$-e_{3}$	e_{31}	$-e_{23}$	-1	$-e_{43}$	$-e_{2}$	e_{1}	e_{321}	$-e_{4}$	$-e_{234}$	e_{314}	e_{12}	$\mathbb{1}$	e_{41}	$-e_{42}$	e_{124}
e_{314}	$-e_{2}$	$-e_{12}$	-1	e_{23}	$-e_{42}$	e_{3}	e_{321}	$-e_{1}$	e_{234}	$-e_{4}$	$-e_{124}$	e_{31}	$-e_{41}$	$\mathbb{1}$	e_{43}	e_{314}
e_{234}	$-e_{1}$	-1	e_{12}	$-e_{31}$	$-e_{41}$	e_{321}	$-e_{3}$	e_{2}	$-e_{314}$	e_{124}	$-e_{4}$	e_{23}	e_{42}	$-e_{43}$	$\mathbb{1}$	e_{234}
$\mathbb{1}$	1	e_{1}	e_{2}	e_{3}	e_{4}	e_{23}	e_{31}	e_{12}	e_{43}	e_{42}	e_{41}	e_{321}	e_{124}	e_{314}	e_{234}	$\mathbb{1}$

Geometric Product and Antiproduct



Geometric Product and Antiproduct

- $\mathbf{1}$ is the multiplicative identity of the product

$$\mathbf{1} \mathbin{\dot\wedge} \mathbf{a} = \mathbf{a} \mathbin{\dot\wedge} \mathbf{1} = \mathbf{a}$$

- $\mathbb{1}$ is the multiplicative identity of the antiproduct

$$\mathbb{1} \mathbin{\dot\vee} \mathbf{a} = \mathbf{a} \mathbin{\dot\vee} \mathbb{1} = \mathbf{a}$$

Reverse

- Unary operation called “reverse” rearranges vector basis element factors so they’re multiplied in reverse order

- If this results in an odd permutation, then the effect is that the term is negated

- Mechanism underlying conjugate operation

Antireverse

- As with everything in GA, the reverse has a dual operation, the “antireverse”
- The antireverse rearranges factors so that antivector basis elements are multiplied in reverse order under the antiproduct

Reverses

- The reverse of $\mathbf{a}$ is written $\tilde{\mathbf{a}}$

- The antireverse of $\mathbf{a}$ is written $\underset{\sim}{\mathbf{a}}$

Basis element $\mathbf{a}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_{23}$	$\mathbf{e}_{31}$	$\mathbf{e}_{12}$	$\mathbf{e}_{43}$	$\mathbf{e}_{42}$	$\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{124}$	$\mathbf{e}_{314}$	$\mathbf{e}_{234}$	$\mathbb{1}$
Reverse $\tilde{\mathbf{a}}$	$\mathbf{1}$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$-\mathbf{e}_{321}$	$-\mathbf{e}_{124}$	$-\mathbf{e}_{314}$	$-\mathbf{e}_{234}$	$\mathbb{1}$
Antireverse $\underset{\sim}{\mathbf{a}}$	$\mathbf{1}$	$-\mathbf{e}_1$	$-\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_4$	$-\mathbf{e}_{23}$	$-\mathbf{e}_{31}$	$-\mathbf{e}_{12}$	$-\mathbf{e}_{43}$	$-\mathbf{e}_{42}$	$-\mathbf{e}_{41}$	$\mathbf{e}_{321}$	$\mathbf{e}_{124}$	$\mathbf{e}_{314}$	$\mathbf{e}_{234}$	$\mathbb{1}$

Reverses and Complements

- In GA, multiplication by $\mathbb{1}$ has long been used to algebraically calculate a dual
- This doesn't work in PGA because $\mathbf{e}_4 ⟑ \mathbf{e}_4 = 0$

- With only one product, only part of the dual gets calculated:

$$\overline{\mathbf{a}_{\bullet}} = \tilde{\mathbf{a}} ⟑ \mathbb{1}$$

$$\underline{\mathbf{a}_{\bullet}} = \underset{\sim}{\mathbf{a}} ⟑ \mathbb{1}$$

Reverses and Complements

- The antiproduct is necessary for the remaining pieces of the dual:

$$\overline{\mathbf{a}_{\circ}} = \mathbf{1} \mathbin{\dot{\vee}} \tilde{\mathbf{a}}$$

$$\underline{\mathbf{a}_{\circ}} = \mathbf{1} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{a}}$$

- Complete duals can now be written as

$$\overline{\mathbf{a}} = \tilde{\mathbf{a}} \mathbin{\dot{\wedge}} \mathbb{1} + \mathbf{1} \mathbin{\dot{\vee}} \tilde{\mathbf{a}}$$

$$\underline{\mathbf{a}} = \underset{\sim}{\mathbf{a}} \mathbin{\dot{\wedge}} \mathbb{1} + \mathbf{1} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{a}}$$

Reflection Through Plane

- All isometries can be broken down into reflections through one or more planes

- Isometries fall into two classes
  - Even number of reflections: proper isometry
  - Odd number of reflections: improper isometry

Fundamental Operation

- Remember, all objects are two things at once

- Reflect dual point of $\mathbf{f}$ through dual plane of $\mathbf{p}$:

$$-\mathbf{p} \mathbin{\dot{\wedge}} \mathbf{f} \mathbin{\dot{\wedge}} \tilde{\mathbf{p}}$$

- Reflect point $\mathbf{p}$ through plane $\mathbf{f}$:

$$-\mathbf{f} \mathbin{\dot{\vee}} \mathbf{p} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{f}}$$

Fundamental Operation

- We can choose to identify objects by the dimensions that are absent/empty and use the geometric product

- Or we can choose to identify objects by the dimensions that are present/full and use the geometric antiproduct

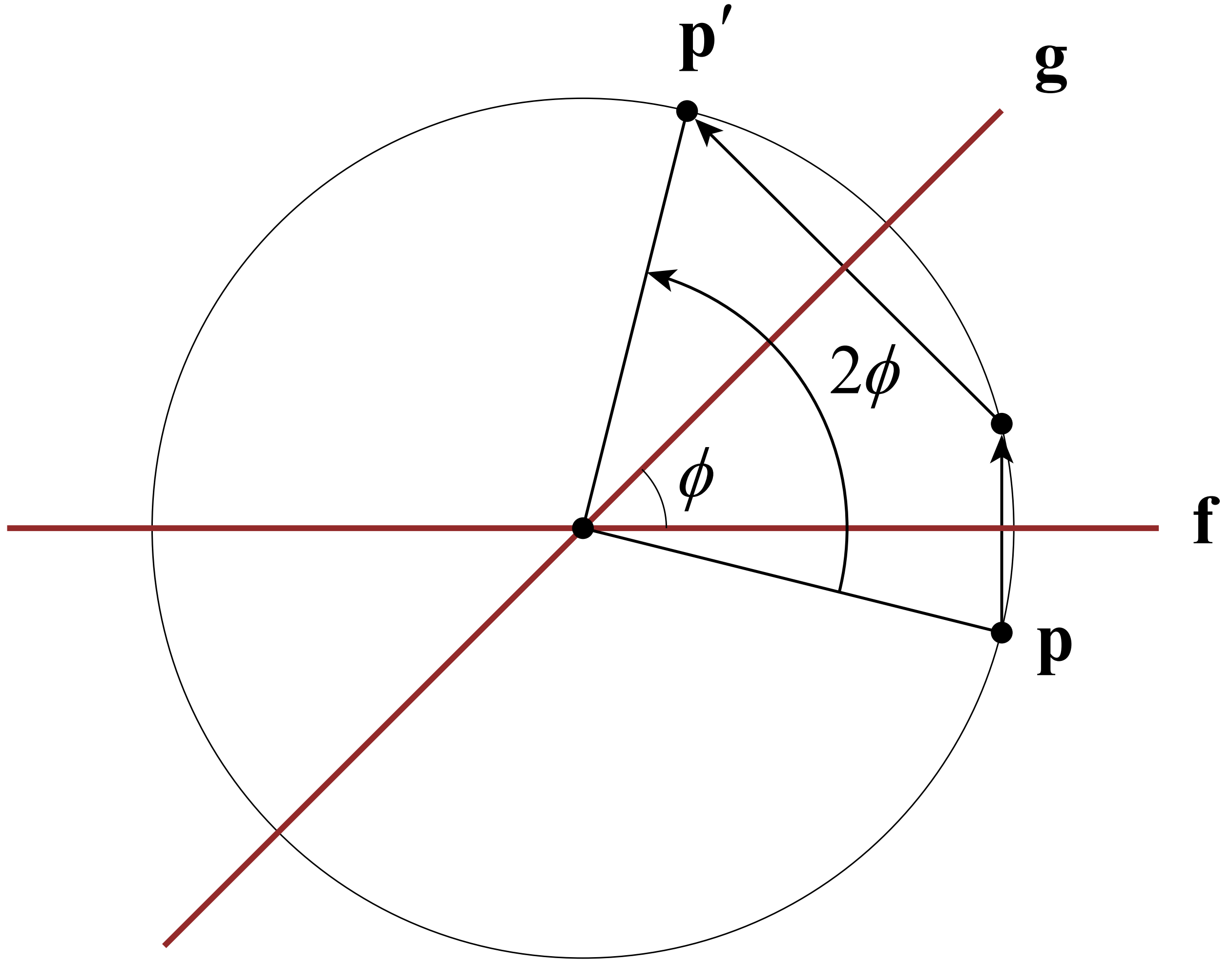
Fundamental Operation

- Both methods are equally valid and produce the same results
- We choose the second option so that points, lines, and planes remain 1, 2, and 3 dimensional in projective space, respectively

Reflection Through Two Planes

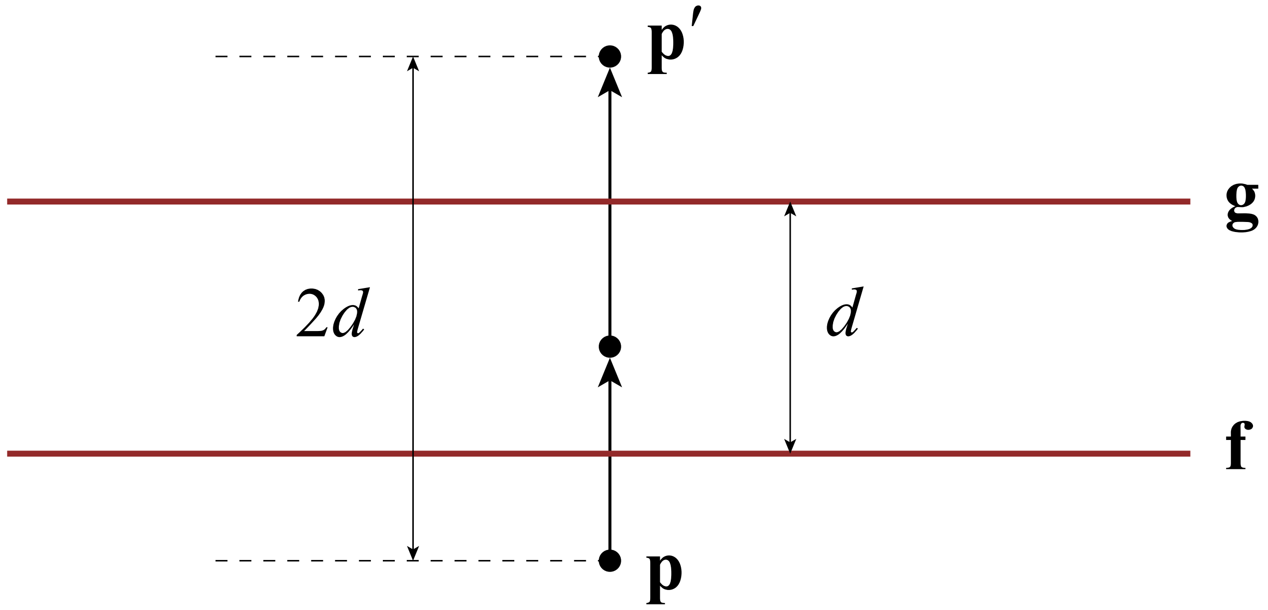
- Reflection through two planes meeting at an angle ϕ

- Rotates about line of intersection by 2ϕ



Reflection Through Two Planes

- If planes are parallel, result is a translation



Motors

- Operator that performs a general proper Euclidean isometry
  - Any combination of rotations and translations
  - Product of an even number of reflections

- Portmanteau of “motion operator” or “moment vector”

Motors

- General form:

$$\mathbf{Q} = \underbrace{r_x \mathbf{e}_{41} + r_y \mathbf{e}_{42} + r_z \mathbf{e}_{43} + r_w \mathbb{1}}_{\text{Rotation}} + u_x \mathbf{e}_{23} + u_y \mathbf{e}_{31} + u_z \mathbf{e}_{12} + u_w$$

- Transformation: $\mathbf{a}' = \mathbf{Q} \mathbin{\dot{\vee}} \mathbf{a} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{Q}}$

Flectors

- Operator that performs a general improper Euclidean isometry
  - Any combination of a reflection with other rotations and translations
  - Product of an odd number of reflections

- Portmanteau of “reflection operator”

Flectors

- General form:

$$\mathbf{G} = \underbrace{s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_4}_{\text{Point}} + \underbrace{h_x\mathbf{e}_{234} + h_y\mathbf{e}_{314} + h_z\mathbf{e}_{124} + h_w\mathbf{e}_{321}}_{\text{Plane}}$$

- Transformation: $\mathbf{a}' = -\mathbf{G} \mathbin{\dot\vee} \mathbf{a} \mathbin{\dot\vee} \underset{\sim}{\mathbf{G}}$

Five Types of Objects in $\mathcal{G}_{3,0,1}$

- Point
  - Four vector components
- Line
  - Six bivector components
- Plane
  - Four trivector components

Five Types of Objects in $\mathcal{G}_{3,0,1}$

- Motor
  - Eight components: scalar, bivector, antiscalar
- Flector
  - Eight components: vector, trivector

Bulk and Weight

- Motors and flectors also have bulk and weight

Type	Definition	Bulk	Weight
Point	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$\mathbf{p}_\bullet = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$	$\mathbf{p}_\circ = p_w\mathbf{e}_4$
Line	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\mathbf{L}_\bullet = m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\mathbf{L}_\circ = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43}$
Plane	$\mathbf{f} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124} + f_w\mathbf{e}_{321}$	$\mathbf{f}_\bullet = f_w\mathbf{e}_{321}$	$\mathbf{f}_\circ = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124}$
Motor	$\mathbf{Q} = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1} + u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$	$\mathbf{Q}_\bullet = u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$	$\mathbf{Q}_\circ = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1}$
Flector	$\mathbf{G} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_4 + h_x\mathbf{e}_{234} + h_y\mathbf{e}_{314} + h_z\mathbf{e}_{124} + h_w\mathbf{e}_{321}$	$\mathbf{G}_\bullet = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + h_w\mathbf{e}_{321}$	$\mathbf{G}_\circ = s_w\mathbf{e}_4 + h_x\mathbf{e}_{234} + h_y\mathbf{e}_{314} + h_z\mathbf{e}_{124}$

Geometric Property

- Not every possible multivector $\mathbf{a}$ is a valid geometric object

- Must satisfy $\mathbf{a} \mathbin{\dot{\wedge}} \tilde{\mathbf{a}} = \text{scalar}$
- Equivalently $\mathbf{a} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{a}} = \text{antiscalar}$

- All vectors and antivectors are valid

Geometric Property

- This imposes the following requirements

Type	Definition	Requirement
Point	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	—
Line	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$v_x m_x + v_y m_y + v_z m_z = 0$
Plane	$\mathbf{f} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124} + f_w\mathbf{e}_{321}$	—
Motor	$\mathbf{Q} = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1} + u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$	$r_x u_x + r_y u_y + r_z u_z + r_w u_w = 0$
Flector	$\mathbf{G} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_4 + h_x\mathbf{e}_{234} + h_y\mathbf{e}_{314} + h_z\mathbf{e}_{124} + h_w\mathbf{e}_{321}$	$s_x h_x + s_y h_y + s_z h_z + s_w h_w = 0$

Norms

- Since there are two geometric products, there are two different norms

- Bulk norm: $$\|\mathbf{a}\|_\bullet = \sqrt{\mathbf{a} \mathbin{\dot{\wedge}} \tilde{\mathbf{a}}}$$

- Weight norm: $$\|\mathbf{a}\|_\circ = \sqrt{\mathbf{a} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{a}}}$$

Norms

- Bulk norm is a scalar
- Weight norm is an antiscalar

Type	Definition	Bulk Norm	Weight Norm
Point	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$\lVert\mathbf{p}\rVert_\bullet = \sqrt{p_x^2 + p_y^2 + p_z^2}$	$\lVert\mathbf{p}\rVert_\circ = \lvert p_w \rvert \mathbb{1}$
Line	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\lVert\mathbf{L}\rVert_\bullet = \sqrt{m_x^2 + m_y^2 + m_z^2}$	$\lVert\mathbf{L}\rVert_\circ = \mathbb{1}\sqrt{v_x^2 + v_y^2 + v_z^2}$
Plane	$\mathbf{f} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124} + f_w\mathbf{e}_{321}$	$\lVert\mathbf{f}\rVert_\bullet = \lvert f_w \rvert$	$\lVert\mathbf{f}\rVert_\circ = \mathbb{1}\sqrt{f_x^2 + f_y^2 + f_z^2}$
Motor	$\mathbf{Q} = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1} + u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$	$\lVert\mathbf{Q}\rVert_\bullet = \sqrt{u_x^2 + u_y^2 + u_z^2 + u_w^2}$	$\lVert\mathbf{Q}\rVert_\circ = \mathbb{1}\sqrt{r_x^2 + r_y^2 + r_z^2 + r_w^2}$
Flector	$\mathbf{G} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_4 + h_x\mathbf{e}_{234} + h_y\mathbf{e}_{314} + h_z\mathbf{e}_{124} + h_w\mathbf{e}_{321}$	$\lVert\mathbf{G}\rVert_\bullet = \sqrt{s_x^2 + s_y^2 + s_z^2 + h_w^2}$	$\lVert\mathbf{G}\rVert_\circ = \mathbb{1}\sqrt{h_x^2 + h_y^2 + h_z^2 + s_w^2}$

Unitization

- An object is unitized when its weight norm is $\mathbb{1}$
- This happens when the coefficients satisfy the following conditions

Type	Definition	Unitization
Point	$\mathbf{p} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4$	$p_w^2 = 1$
Line	$\mathbf{L} = v_x \mathbf{e}_{41} + v_y \mathbf{e}_{42} + v_z \mathbf{e}_{43} + m_x \mathbf{e}_{23} + m_y \mathbf{e}_{31} + m_z \mathbf{e}_{12}$	$v_x^2 + v_y^2 + v_z^2 = 1$
Plane	$\mathbf{f} = f_x \mathbf{e}_{234} + f_y \mathbf{e}_{314} + f_z \mathbf{e}_{124} + f_w \mathbf{e}_{321}$	$f_x^2 + f_y^2 + f_z^2 = 1$
Motor	$\mathbf{Q} = r_x \mathbf{e}_{41} + r_y \mathbf{e}_{42} + r_z \mathbf{e}_{43} + r_w \mathbb{1} + u_x \mathbf{e}_{23} + u_y \mathbf{e}_{31} + u_z \mathbf{e}_{12} + u_w$	$r_x^2 + r_y^2 + r_z^2 + r_w^2 = 1$
Flector	$\mathbf{G} = s_x \mathbf{e}_1 + s_y \mathbf{e}_2 + s_z \mathbf{e}_3 + s_w \mathbf{e}_4 + h_x \mathbf{e}_{234} + h_y \mathbf{e}_{314} + h_z \mathbf{e}_{124} + h_w \mathbf{e}_{321}$	$s_w^2 + h_x^2 + h_y^2 + h_z^2 = 1$

Homogeneous Magnitude

- What do the norms represent?

- We add them together and get a scalar/antiscalar pair $x\mathbf{1} + y\mathbb{1}$

- This is a homogeneous magnitude that has a bulk and a weight and can also be unitized!

Homogeneous Magnitude

Type	Definition	Homogeneous Magnitude
Point	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$\lVert\mathbf{p}\rVert = \sqrt{p_x^2 + p_y^2 + p_z^2} + \lvert p_w \rvert \mathbb{1}$
Line	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\lVert\mathbf{L}\rVert = \sqrt{m_x^2 + m_y^2 + m_z^2} + \mathbb{1}\sqrt{v_x^2 + v_y^2 + v_z^2}$
Plane	$\mathbf{f} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124} + f_w\mathbf{e}_{321}$	$\lVert\mathbf{f}\rVert = \lvert f_w \rvert + \mathbb{1}\sqrt{f_x^2 + f_y^2 + f_z^2}$
Motor	$\mathbf{Q} = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1} + u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$	$\lVert\mathbf{Q}\rVert = \sqrt{u_x^2 + u_y^2 + u_z^2 + u_w^2} + \mathbb{1}\sqrt{r_x^2 + r_y^2 + r_z^2 + r_w^2}$
Flector	$\mathbf{G} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_4 + h_x\mathbf{e}_{234} + h_y\mathbf{e}_{314} + h_z\mathbf{e}_{124} + h_w\mathbf{e}_{321}$	$\lVert\mathbf{G}\rVert = \sqrt{s_x^2 + s_y^2 + s_z^2 + h_w^2} + \mathbb{1}\sqrt{h_x^2 + h_y^2 + h_z^2 + s_w^2}$

Geometric Norm

- The geometric norm is produced by unitizing the homogeneous magnitude so that its weight (the antiscalar part) is just $\mathbb{1}$
- This gives us a concrete measurement of Euclidean distance

Geometric Norm

Type	Definition	Geometric Norm	Interpretation
Point	$\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3 + p_w\mathbf{e}_4$	$\left\|\widehat{\mathbf{p}}\right\| = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$	Distance from the origin to the point **p**. Half the distance that the origin is moved by the flector **p**.
Line	$\mathbf{L} = v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43} + m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12}$	$\left\|\widehat{\mathbf{L}}\right\| = \sqrt{\dfrac{m_x^2 + m_y^2 + m_z^2}{v_x^2 + v_y^2 + v_z^2}}$	Perpendicular distance from the origin to the line **L**. Half the distance that the origin is moved by the motor **L**.
Plane	$\mathbf{f} = f_x\mathbf{e}_{234} + f_y\mathbf{e}_{314} + f_z\mathbf{e}_{124} + f_w\mathbf{e}_{321}$	$\left\|\widehat{\mathbf{f}}\right\| = \dfrac{\lvert f_w \rvert}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$	Perpendicular distance from the origin to the plane **f**. Half the distance that the origin is moved by the flector **f**.
Motor	$\mathbf{Q} = r_x\mathbf{e}_{41} + r_y\mathbf{e}_{42} + r_z\mathbf{e}_{43} + r_w\mathbb{1} + u_x\mathbf{e}_{23} + u_y\mathbf{e}_{31} + u_z\mathbf{e}_{12} + u_w$	$\left\|\widehat{\mathbf{Q}}\right\| = \sqrt{\dfrac{u_x^2 + u_y^2 + u_z^2 + u_w^2}{r_x^2 + r_y^2 + r_z^2 + r_w^2}}$	Half the distance that the origin is moved by the motor **Q**.
Flector	$\mathbf{G} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + s_w\mathbf{e}_4 + h_x\mathbf{e}_{234} + h_y\mathbf{e}_{314} + h_z\mathbf{e}_{124} + h_w\mathbf{e}_{321}$	$\left\|\widehat{\mathbf{G}}\right\| = \sqrt{\dfrac{s_x^2 + s_y^2 + s_z^2 + h_w^2}{h_x^2 + h_y^2 + h_z^2 + s_w^2}}$	Half the distance that the origin is moved by the flector **G**.

Commutators

- Four different commutators
- Combining addition or subtraction with geometric product or antiproduct

$$[\mathbf{a},\mathbf{b}]^{⟑}_{-} = \frac{1}{2}\left(\mathbf{a} ⟑ \tilde{\mathbf{b}} - \mathbf{b} ⟑ \tilde{\mathbf{a}}\right) \qquad [\mathbf{a},\mathbf{b}]^{⟇}_{-} = \frac{1}{2}\left(\mathbf{a} ⟇ \tilde{\mathbf{b}} - \mathbf{b} ⟇ \tilde{\mathbf{a}}\right)$$

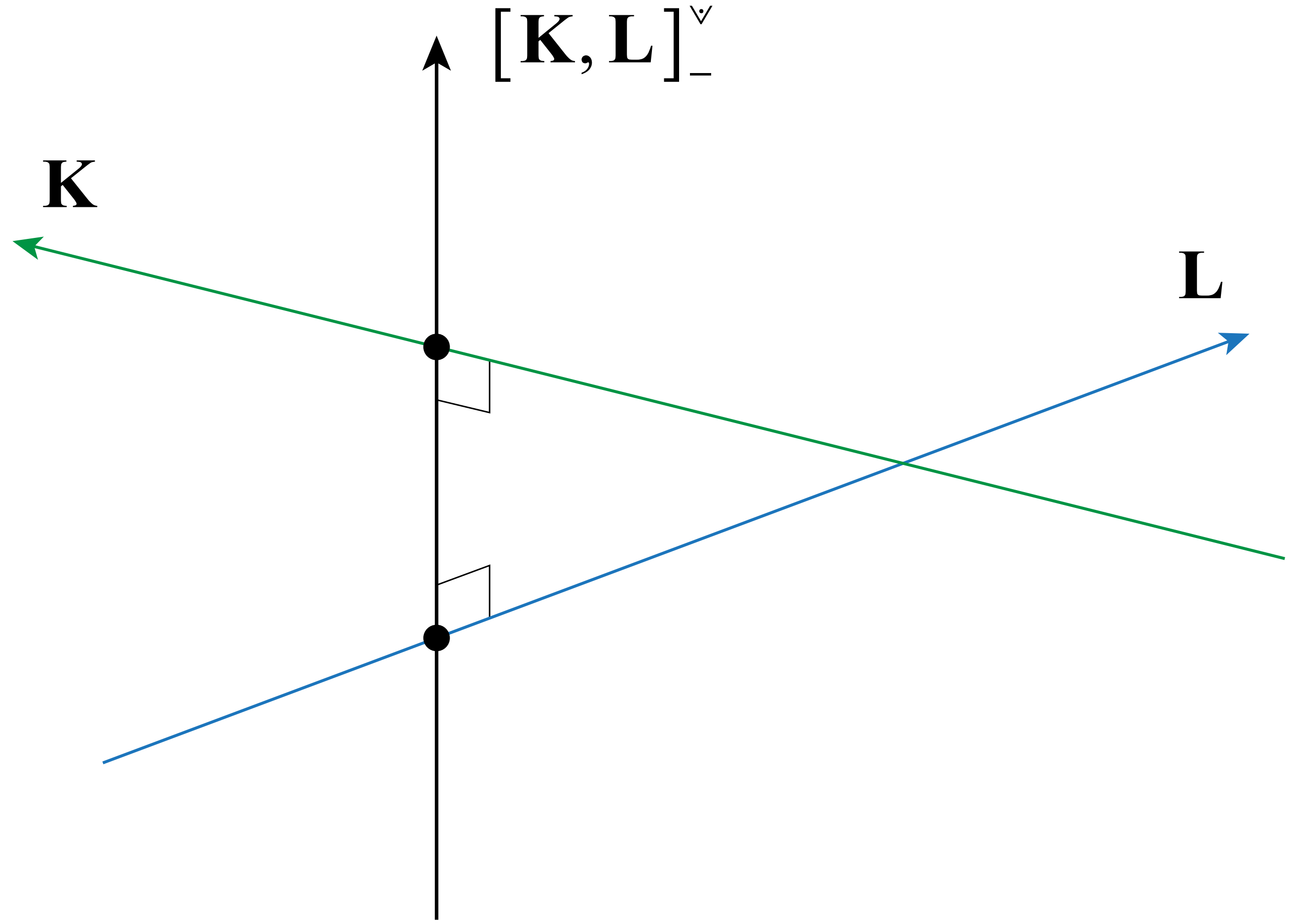
$$[\mathbf{a},\mathbf{b}]^{⟑}_{+} = \frac{1}{2}\left(\mathbf{a} ⟑ \tilde{\mathbf{b}} + \mathbf{b} ⟑ \tilde{\mathbf{a}}\right) \qquad [\mathbf{a},\mathbf{b}]^{⟇}_{+} = \frac{1}{2}\left(\mathbf{a} ⟇ \tilde{\mathbf{b}} + \mathbf{b} ⟇ \tilde{\mathbf{a}}\right)$$

Commutators

- All of the join and meet operations can be done with commutators, but there’s more...
- A commutator can construct the line between two lines
- Commutators also give Euclidean distances between different objects

Line Between Two Lines

$$[\mathbf{K},\mathbf{L}]_{-}^{\vee} = (v_y w_z - v_z w_y)\mathbf{e}_{41} + (v_z w_x - v_x w_z)\mathbf{e}_{42} + (v_x w_y - v_y w_x)\mathbf{e}_{43}$$
$$+ (v_y n_z - v_z n_y + m_y w_z - m_z w_y)\mathbf{e}_{23} + (v_z n_x - v_x n_z + m_z w_x - m_x w_z)\mathbf{e}_{31} + (v_x n_y - v_y n_x + m_x w_y - m_y w_x)\mathbf{e}_{12}$$



Euclidean Distances

Formula	**Interpretation**
$\dfrac{\left\Vert[\mathbf{p},\mathbf{q}]^{\wedge}_{-}\right\Vert_{\bigcirc}}{\left\Vert[\mathbf{p},\mathbf{q}]^{\vee}_{+}\right\Vert_{\bigcirc}} = \dfrac{\sqrt{(q_x p_w - p_x q_w)^2 + (q_y p_w - p_y q_w)^2 + (q_z p_w - p_z q_w)^2}}{\lvert p_w q_w \rvert}$	Distance between points **p** and **q**.
$\dfrac{\left\Vert[\mathbf{p},\mathbf{L}]^{\wedge}_{-}\right\Vert_{\bigcirc}}{\left\Vert[\mathbf{p},\mathbf{L}]^{\vee}_{+}\right\Vert_{\bigcirc}} = \dfrac{\sqrt{(v_y p_z - v_z p_y + m_x p_w)^2 + (v_z p_x - v_x p_z + m_y p_w)^2 + (v_x p_y - v_y p_x + m_z p_w)^2}}{\lvert p_w \rvert \sqrt{v_x^2 + v_y^2 + v_z^2}}$	Perpendicular distance between point **p** and line **L**.
$\dfrac{\left\Vert[\mathbf{p},\mathbf{f}]^{\wedge}_{+}\right\Vert_{\bigcirc}}{\left\Vert[\mathbf{p},\mathbf{f}]^{\vee}_{-}\right\Vert_{\bigcirc}} = \dfrac{\lvert p_x f_x + p_y f_y + p_z f_z + p_w f_w \rvert}{\lvert p_w \rvert \sqrt{f_x^2 + f_y^2 + f_z^2}}$	Perpendicular distance between point **p** and plane **f**.
$\dfrac{\left\Vert[\mathbf{K},\mathbf{L}]^{\wedge}_{+}\right\Vert_{\bigcirc}}{\left\Vert[\mathbf{K},\mathbf{L}]^{\vee}_{-}\right\Vert_{\bigcirc}} = \dfrac{\lvert v_x n_x + v_y n_y + v_z n_z + w_x m_x + w_y m_y + w_z m_z \rvert}{\sqrt{(v_y w_z - v_z w_y)^2 + (v_z w_x - v_x w_z)^2 + (v_x w_y - v_y w_x)^2}}$	Perpendicular distance between lines **K** and **L**.

Rotation

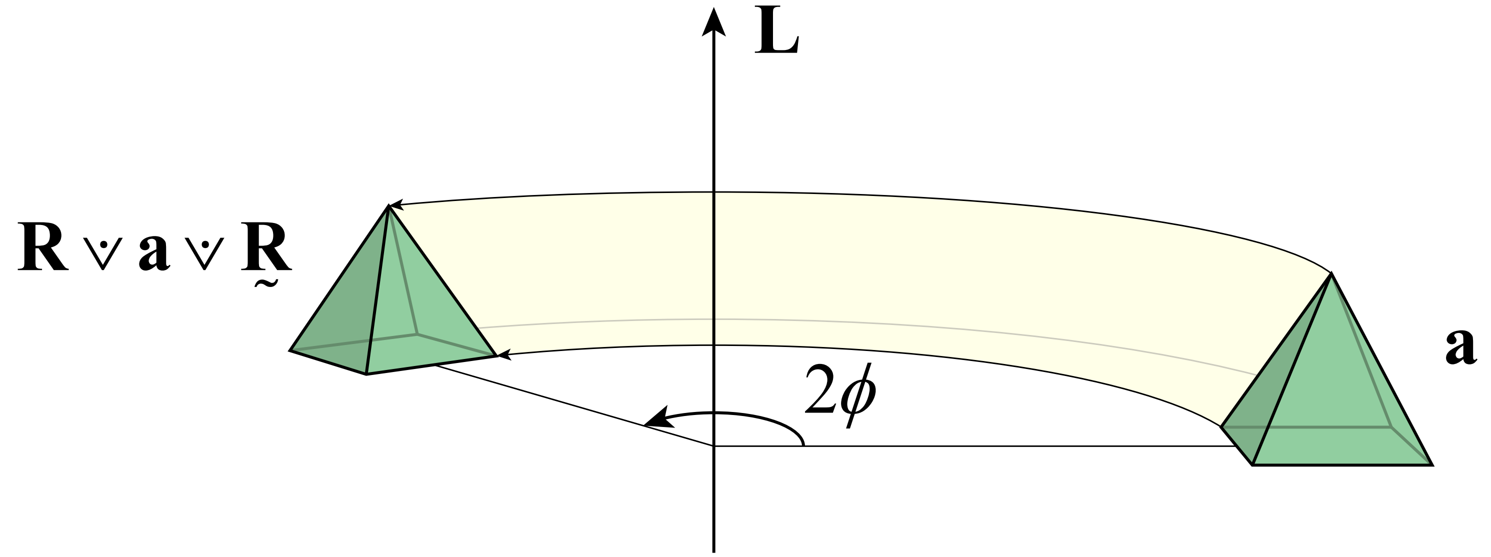
- Reflect through plane $\mathbf{f}$ and then plane $\mathbf{g}$

$$\begin{aligned}\mathbf{g} \,\dot{\vee}\, \mathbf{f} &= \left(f_y g_z - f_z g_y\right)\mathbf{e}_{41} + \left(f_z g_x - f_x g_z\right)\mathbf{e}_{42} + \left(f_x g_y - f_y g_x\right)\mathbf{e}_{43} \\ &+ \left(f_w g_x - f_x g_w\right)\mathbf{e}_{23} + \left(f_w g_y - f_y g_w\right)\mathbf{e}_{31} + \left(f_w g_z - f_z g_w\right)\mathbf{e}_{12} \\ &+ \left(f_x g_x + f_y g_y + f_z g_z\right)\mathbb{1}\end{aligned}$$

- Contains line where planes intersect
- Also contains angle information

Rotation

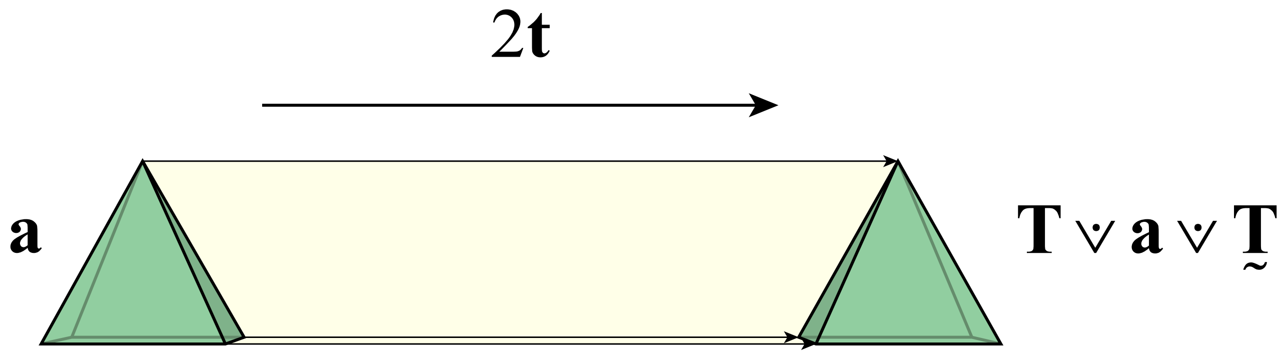
- Rotate about line $\mathbf{L}$ through angle 2ϕ



$$\mathbf{R} = \mathbf{L}\sin\phi + \mathbb{1}\cos\phi$$

Translation

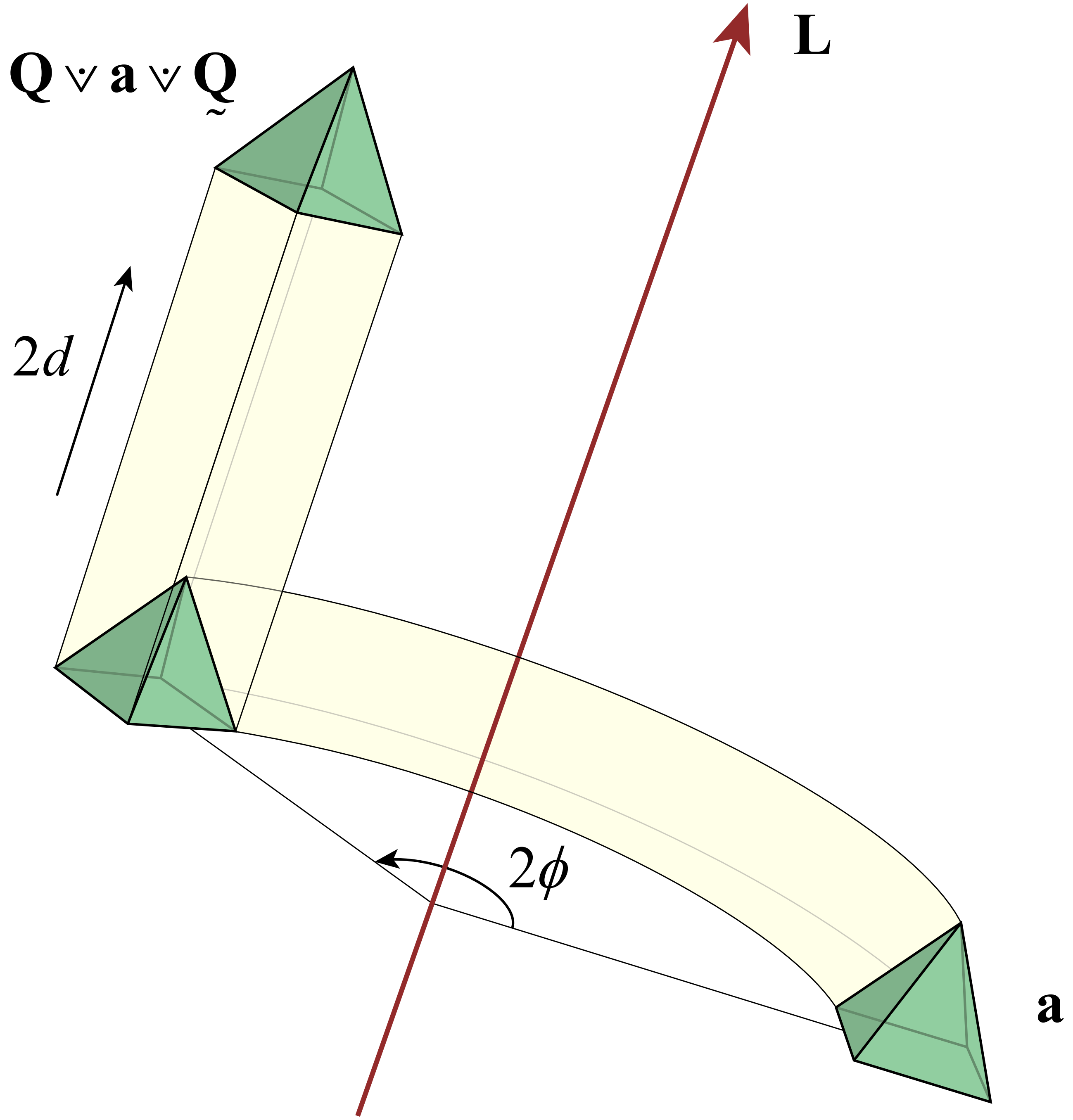
- Parallel planes intersect at a line at infinity
- And angle between them is zero



$$\mathbf{T} = t_x \mathbf{e}_{23} + t_y \mathbf{e}_{31} + t_z \mathbf{e}_{12} + \mathbb{1}$$

Motor

- All proper 3D isometries can be described as a screw motion
- A rotation about a line and a displacement along the same line
- General form of motor



$$\mathbf{Q} = \mathbf{L}\sin\phi + \mathbb{1}\cos\phi + (d \vee \mathbf{L})\cos\phi - d\sin\phi$$

Motors from Geometries

Motor	Description
$\mathbf{g} \vee \mathbf{f} = (f_y g_z - f_z g_y)\mathbf{e}_{41} + (f_z g_x - f_x g_z)\mathbf{e}_{42} + (f_x g_y - f_y g_x)\mathbf{e}_{43} + (f_w g_x - f_x g_w)\mathbf{e}_{23} + (f_w g_y - f_y g_w)\mathbf{e}_{31} + (f_w g_z - f_z g_w)\mathbf{e}_{12} + (f_x g_x + f_y g_y + f_z g_z)\mathbb{1}$	Rotation about the line where planes **f** and **g** intersect by twice the angle between them in the direction from **f** to **g**.
$\mathbf{L} \vee \mathbf{K} = (v_y w_z - v_z w_y)\mathbf{e}_{41} + (v_z w_x - v_x w_z)\mathbf{e}_{42} + (v_x w_y - v_y w_x)\mathbf{e}_{43} + (v_y n_z - v_z n_y)\mathbf{e}_{23} + (v_z n_x - v_x n_z)\mathbf{e}_{31} + (v_x n_y - v_y n_x)\mathbf{e}_{12} - (w_y m_z - w_z m_y)\mathbf{e}_{23} - (w_z m_x - w_x m_z)\mathbf{e}_{31} - (w_x m_y - w_y m_x)\mathbf{e}_{12} - (v_x n_x + v_y n_y + v_z n_z) - (w_x m_x + w_y m_y + w_z m_z) - (v_x w_x + v_y w_y + v_z w_z)\mathbb{1}$ $\mathbf{L} = \{\mathbf{v} \mid \mathbf{m}\} \qquad \mathbf{K} = \{\mathbf{w} \mid \mathbf{n}\}$	Rotation about the line containing the closest points on lines **K** and **L** by twice the angle between **v** and **w**. Translation by twice the distance between the lines in the direction from **K** to **L**.
$\mathbf{q} \vee \mathbf{p} = (p_x q_w - q_x p_w)\mathbf{e}_{23} + (p_y q_w - q_y p_w)\mathbf{e}_{31} + (p_z q_w - q_z p_w)\mathbf{e}_{12} - p_w q_w \mathbb{1}$	Translation by twice the distance between points **p** and **q** in the direction from **p** to **q**.

Motor to Matrix

- We eventually want to convert to a 4×4 matrix
- Not as efficient to compute sandwich products a bunch of times
- Let $\mathbf{M}$ be the 4×4 matrix that we would use to transform points

Motor to Matrix

- Define

$$\mathbf{A} = \begin{bmatrix} 1-2\left(r_y^2+r_z^2\right) & 2r_xr_y & 2r_zr_x & 2\left(r_yu_z-r_zu_y\right) \\ 2r_xr_y & 1-2\left(r_z^2+r_x^2\right) & 2r_yr_z & 2\left(r_zu_x-r_xu_z\right) \\ 2r_zr_x & 2r_yr_z & 1-2\left(r_x^2+r_y^2\right) & 2\left(r_xu_y-r_yu_x\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & -2r_zr_w & 2r_yr_w & 2\left(r_wu_x-r_xu_w\right) \\ 2r_zr_w & 0 & -2r_xr_w & 2\left(r_wu_y-r_yu_w\right) \\ -2r_yr_w & 2r_xr_w & 0 & 2\left(r_wu_z-r_zu_w\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Then $\mathbf{M} = \mathbf{A} + \mathbf{B}$ and $\mathbf{M}^{-1} = \mathbf{A} - \mathbf{B}$

Motor Advantages

- Arbitrary rigid motion can be stored as 6 floats

$$\mathbf{Q} = \underbrace{r_x \mathbf{e}_{41} + r_y \mathbf{e}_{42} + r_z \mathbf{e}_{43} + r_w \mathbb{1}}_{\text{Rotation}} + u_x \mathbf{e}_{23} + u_y \mathbf{e}_{31} + u_z \mathbf{e}_{12} + u_w$$

- To be unitized, rotation part has unit length
- Can flip sign to make $r_w \geq 0$
- Then $r_w = \sqrt{1 - r_x^2 - r_y^2 - r_z^2}$

Motor Advantages

- Geometric property requires

$$r_x u_x + r_y u_y + r_z u_z + r_w u_w = 0$$

- Can solve for u_w when other 7 values known

- Not a coincidence that a rigid motion in 3D space has 6 degrees of freedom

Motor Advantages

- Extremely easy to invert: $\mathbf{Q}^{-1} = \tilde{\mathbf{Q}}$
- This just negates the 6 bivector components

- Easier to re-orthogonalize than 4×4 matrix
  - Unitize the weight part $\mathbf{r}$
  - Subtract projection of bulk part $\mathbf{u}$ onto weight part

Motor Interpolation

- Motors interpolate a lot better than matrices
- This is used for dual quaternion skinning

Motor Interpolation

- A motor can be expressed as an exponential with respect to the geometric antiproduct:

$$\mathbf{Q} = e_{\veebar}^{(d+\varphi\mathbb{1})\veebar\mathbf{L}} = \cos_{\veebar}\left(d+\varphi\mathbb{1}\right) + \sin_{\veebar}\left(d+\varphi\mathbb{1}\right)\veebar\mathbf{L}$$

$$\mathbf{Q} = \mathbb{1}\cos\varphi - d\sin\varphi + \left(d \veebar \mathbf{L}\right)\cos\varphi + \mathbf{L}\sin\varphi$$

$$\begin{aligned}\mathbf{Q} = {} & (v_x\mathbf{e}_{41} + v_y\mathbf{e}_{42} + v_z\mathbf{e}_{43})\sin\varphi + \mathbb{1}\cos\varphi - d\sin\varphi \\ & + (dv_x\mathbf{e}_{23} + dv_y\mathbf{e}_{31} + dv_z\mathbf{e}_{12})\cos\varphi \\ & + (m_x\mathbf{e}_{23} + m_y\mathbf{e}_{31} + m_z\mathbf{e}_{12})\sin\varphi\end{aligned}$$

Motor Interpolation

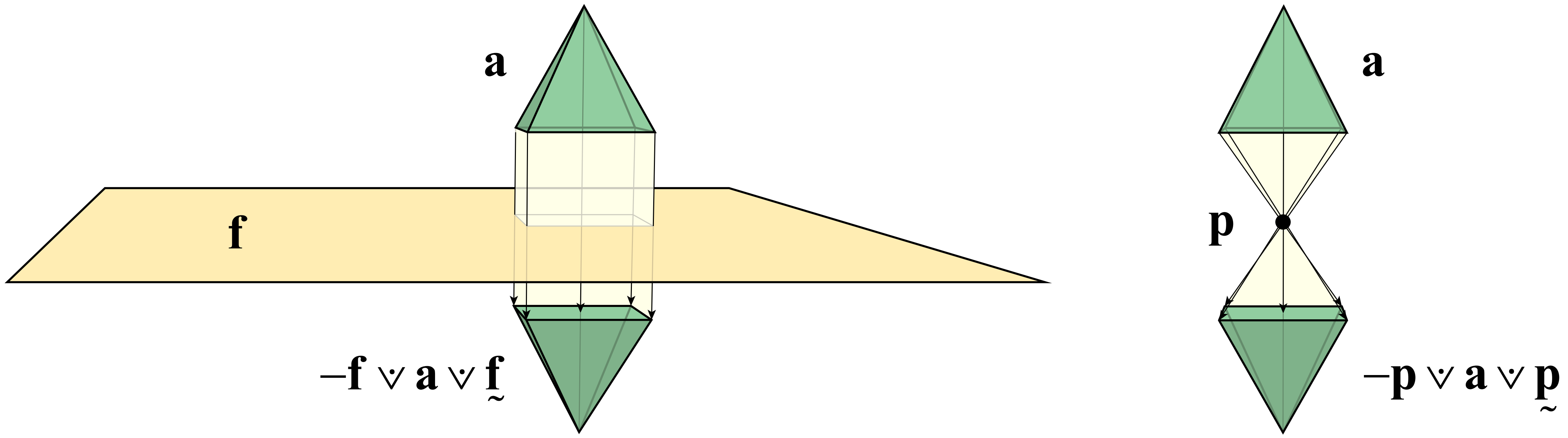
- The exponential form allows for high-quality interpolation, but requires a logarithm

- In practice, linear interpolation and re-unitization are sufficient

Dual Quaternion Skinning

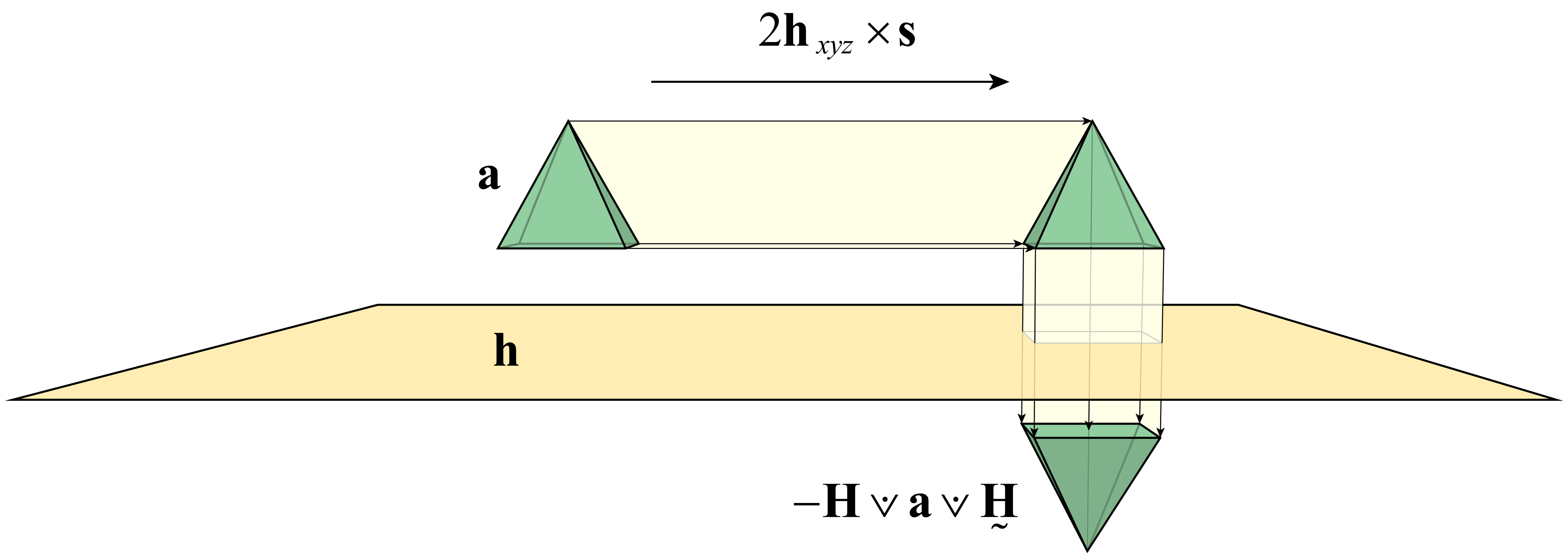
Reflection and Inversion

- Planes and points as isometry operators



Transflection

- A plane and a direction

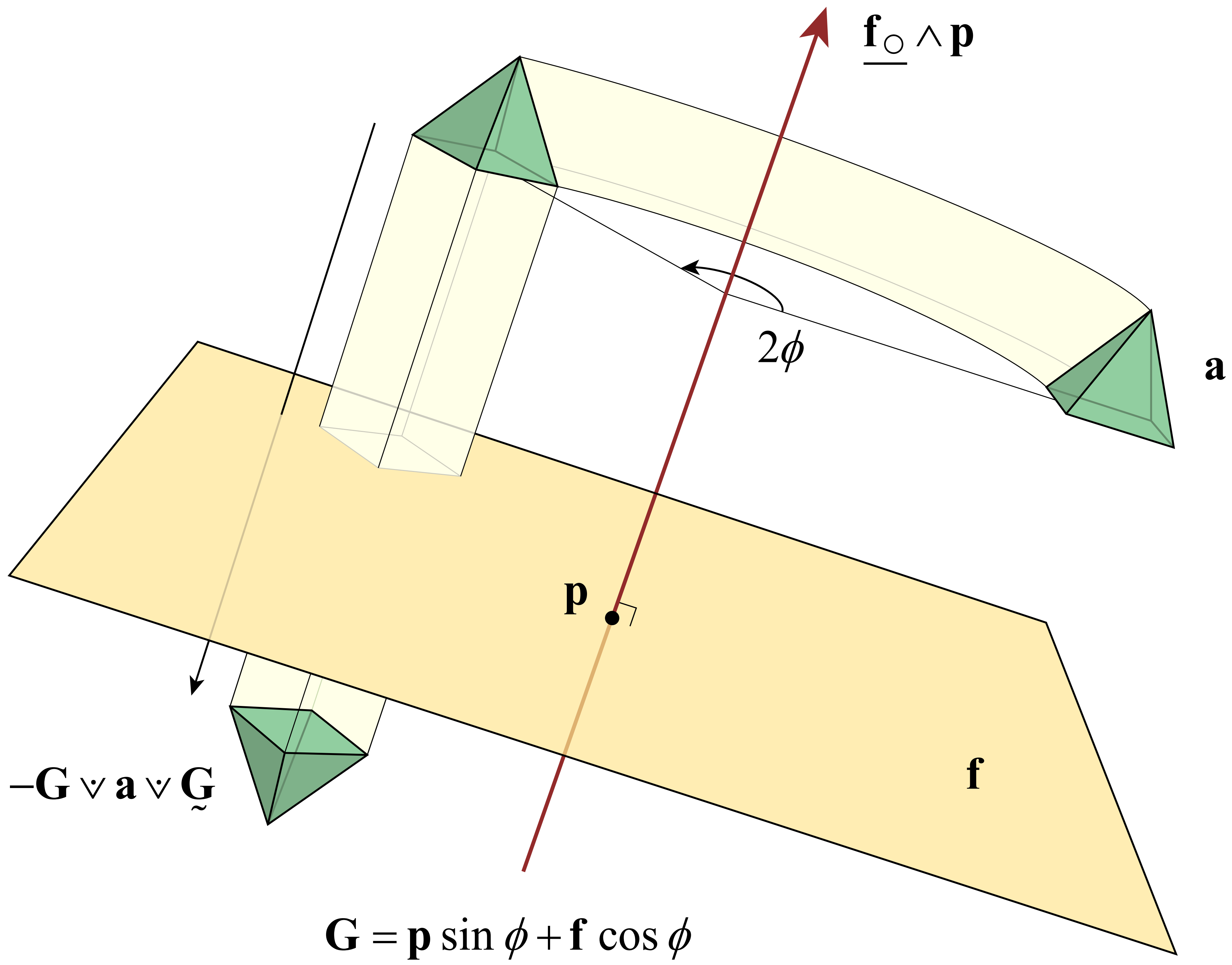


$$\mathbf{H} = s_x\mathbf{e}_1 + s_y\mathbf{e}_2 + s_z\mathbf{e}_3 + \mathbf{h}$$

Flector

- All improper 3D isometries can be described as a rotoreflection
- This is a rotation about a line and a reflection through a plane perpendicular to that line
- General form of a flector

Flector



$\underline{\mathbf{f}_\circ} \wedge \mathbf{p}$
2ϕ
a
p
f
$-\mathbf{G} \vee \mathbf{a} \vee \underset{\sim}{\mathbf{G}}$
$\mathbf{G} = \mathbf{p}\sin\phi + \mathbf{f}\cos\phi$

Flector to Matrix

- Define

$$\mathbf{A} = \begin{bmatrix} 2\left(h_y^2 + h_z^2\right) - 1 & -2h_x h_y & -2h_z h_x & 2\left(s_x s_w - h_x h_w\right) \\ -2h_x h_y & 2\left(h_z^2 + h_x^2\right) - 1 & -2h_y h_z & 2\left(s_y s_w - h_y h_w\right) \\ -2h_z h_x & -2h_y h_z & 2\left(h_x^2 + h_y^2\right) - 1 & 2\left(s_z s_w - h_z h_w\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 2h_z s_w & -2h_y s_w & 2\left(h_y s_z - h_z s_y\right) \\ -2h_z s_w & 0 & 2h_x s_w & 2\left(h_z s_x - h_x s_z\right) \\ 2h_y s_w & -2h_x s_w & 0 & 2\left(h_x s_y - h_y s_x\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Then $\mathbf{M} = \mathbf{A} + \mathbf{B}$ and $\mathbf{M}^{-1} = \mathbf{A} - \mathbf{B}$

More Information

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