Game Math Case Studies

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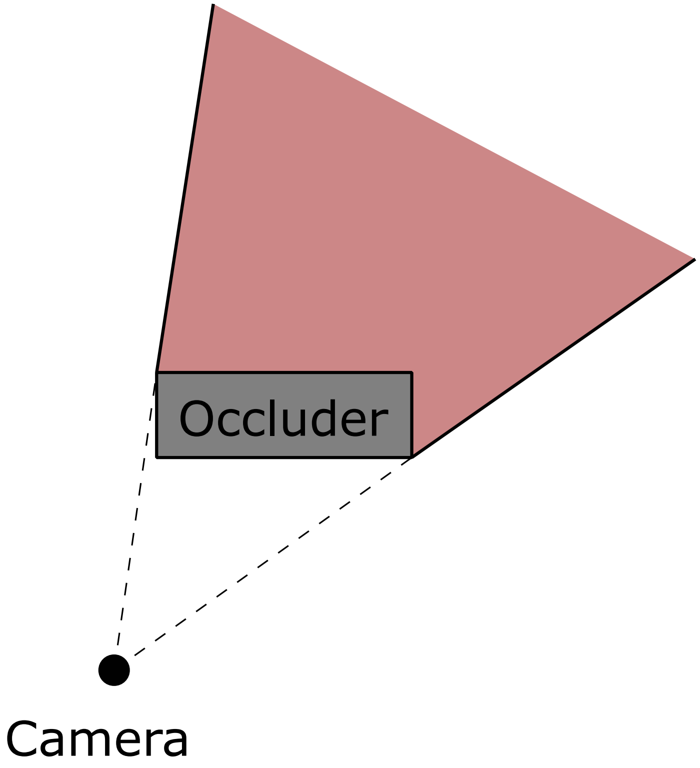
Content of this Talk

- Real-world problems from game dev
  - Small problems, that is, and easy to state
- Actual solutions used in shipping games
  - Using math that's not too advanced
- Strategies for finding elegant solutions

Occlusion boxes

- Plain boxes put in world as occluders
- Extrude away from camera to form occluded region of space where objects don’t need to be rendered
- How to do this most efficiently?


Occluder
Camera

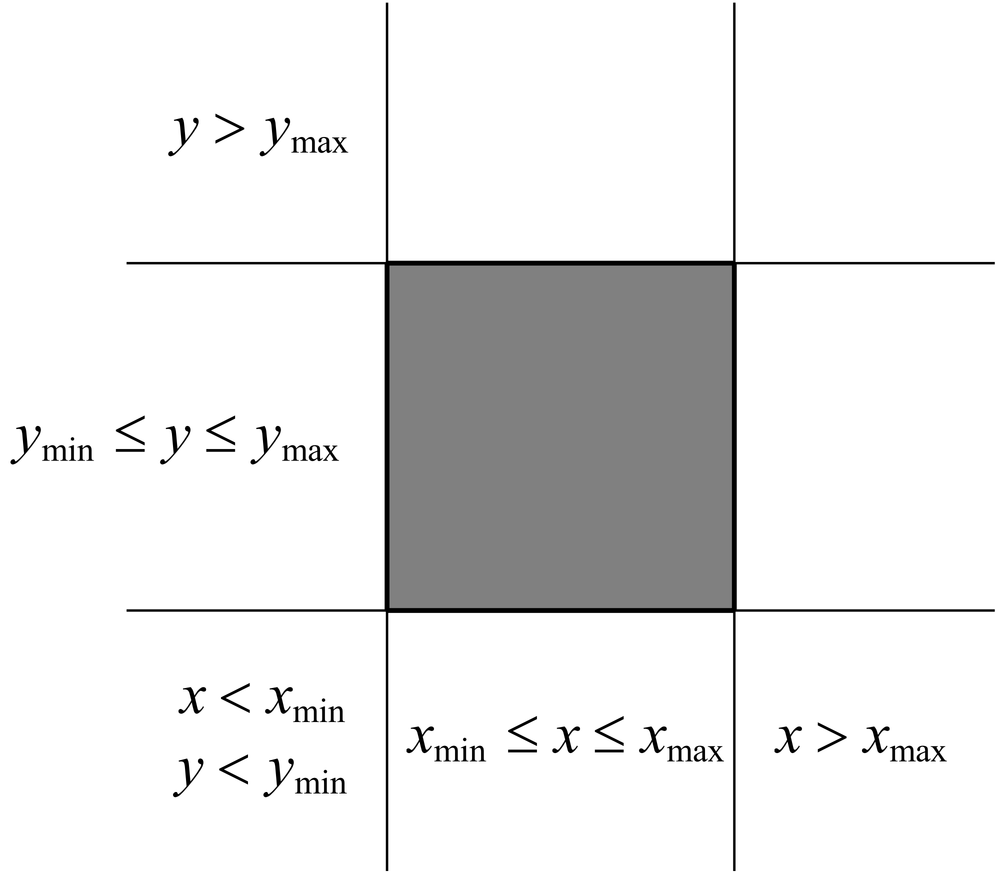
Occlusion boxes

- Could classify box faces as front/back and find silhouette edges
  - Similar to stencil shadow technique

- A better solution accounts for small solution space

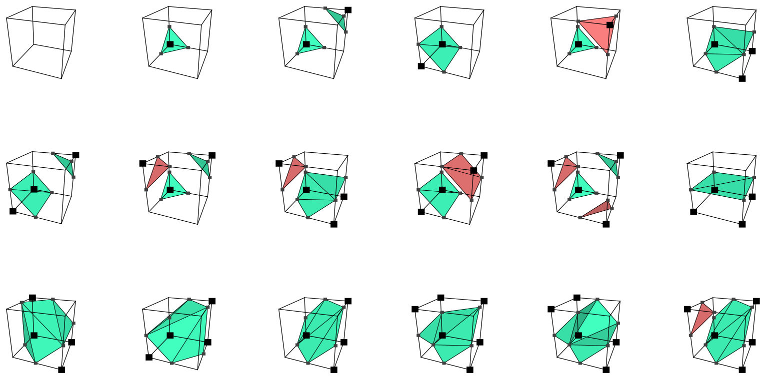
Occlusion boxes

- There are exactly 26 possible silhouettes
- Three possible states for camera position on three different axes
  - position $<$ box min
  - position $>$ box max
  - box min $\leq$ position $\leq$ box max
  - Inside box excluded



condition	code
x > xmax	0x01
x < xmin	0x02
y > ymax	0x04
y < ymin	0x08
z > zmax	0x10
z < zmin	0x20

Finite classifications



Marching Cubes, fixed polarity

(256 cases, 18 classes)

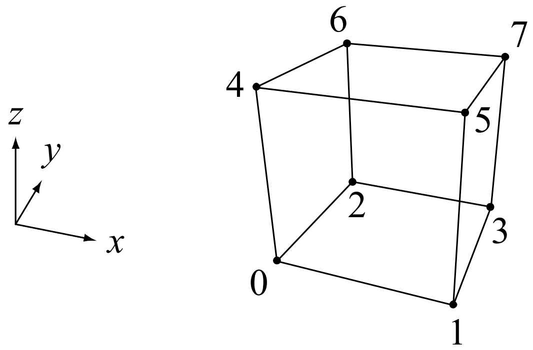
[Transvoxel Algorithm]

(512 cases, 73 classes)

Occlusion boxes

- Calculate camera position state and use table to get silhouette
- Always a closed convex polygon with exactly 4 or 6 vertices and edges

Occlusion boxes



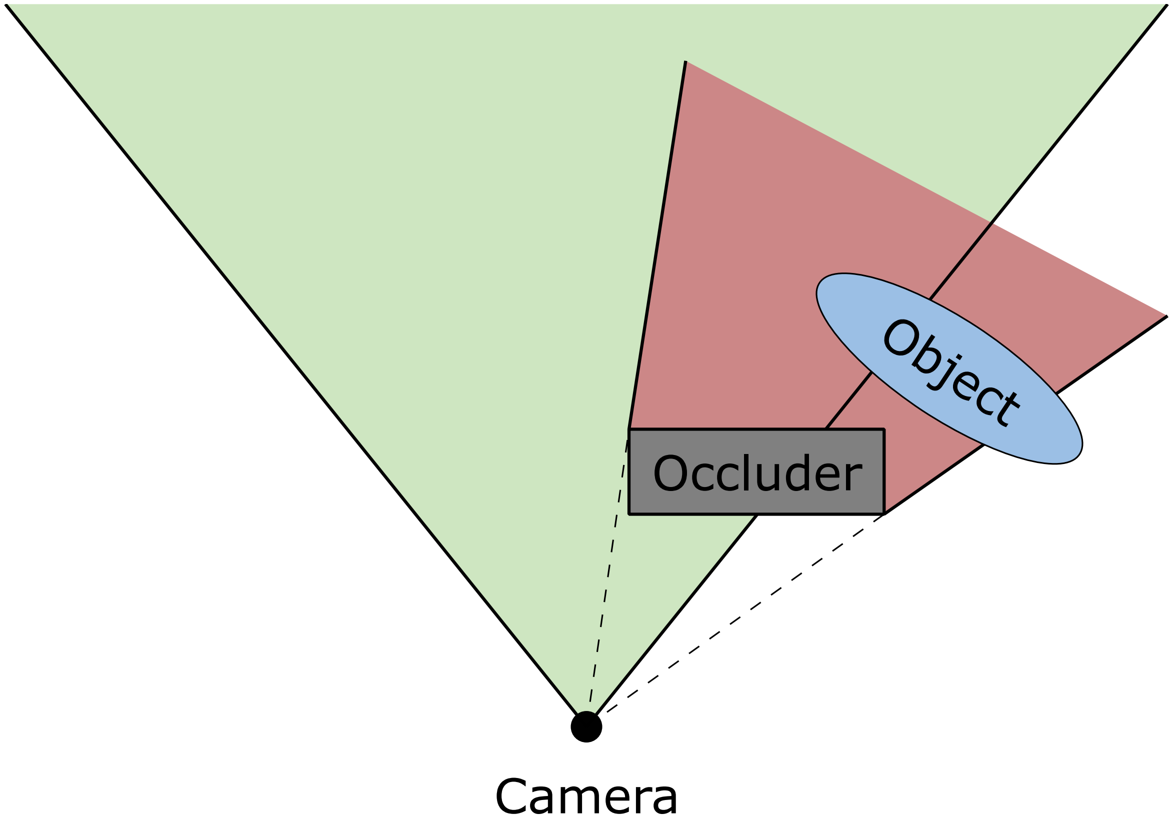

```
// Upper 3 bits = vertex count, lower 5 bits = polygon index
const unsigned_int8 occlusionPolygonIndex[43] =
{
    0x00, 0x80, 0x81, 0x00, 0x82, 0xC9, 0xC8, 0x00, 0x83, 0xC7, 0xC6, 0x00, 0x00, 0x00, 0x00, 0x00,
    0x84, 0xCF, 0xCE, 0x00, 0xD1, 0xD9, 0xD8, 0x00, 0xD0, 0xD7, 0xD6, 0x00, 0x00, 0x00, 0x00, 0x00,
    0x85, 0xCB, 0xCA, 0x00, 0xCD, 0xD5, 0xD4, 0x00, 0xCC, 0xD3, 0xD2
};

// All 26 polygons with vertex indexes from diagram on left
const unsigned_int8 occlusionVertexIndex[26][6] =
{
    {1, 3, 7, 5},
    {2, 0, 4, 6},
    {3, 2, 6, 7},
    {0, 1, 5, 4},
    {4, 5, 7, 6},
    {1, 0, 2, 3},
    {2, 0, 1, 5, 4, 6},
    {0, 1, 3, 7, 5, 4},
    {3, 2, 0, 4, 6, 7},
    {1, 3, 2, 6, 7, 5},
    {1, 0, 4, 6, 2, 3},
    {5, 1, 0, 2, 3, 7},
    {4, 0, 2, 3, 1, 5},
    {0, 2, 6, 7, 3, 1},
    {0, 4, 5, 7, 6, 2},
    {4, 5, 1, 3, 7, 6},
    {1, 5, 7, 6, 4, 0},
    {5, 7, 3, 2, 6, 4},
    {3, 1, 5, 4, 6, 2},
    {2, 3, 7, 5, 4, 0},
    {1, 0, 4, 6, 7, 3},
    {0, 2, 6, 7, 5, 1},
    {7, 6, 2, 0, 1, 5},
    {6, 4, 0, 1, 3, 7},
    {5, 7, 3, 2, 0, 4},
    {4, 5, 1, 3, 2, 6}
};
```

Occlusion boxes

- Any silhouette edge that is off screen can be eliminated to make occlusion region larger
- Gives occluder infinite extent in that direction
- Allows more objects to be occluded because they must be completely inside extruded silhouette to be hidden

Occlusion boxes

- Silhouette edge is culled if both vertices on negative side of some frustum plane
- *And* extruded plane normal and frustum plane normal have positive dot product


Object
Occluder
Camera

Occlusion boxes

- Strategy:

Look for ways to classify solutions

Oblique near plane trick

- Sometimes need a clipping plane for a flat surface in scene
- For example, water or mirror
  - Prevent submerged objects from appearing in reflection

Ordinary frustum

Oblique near plane

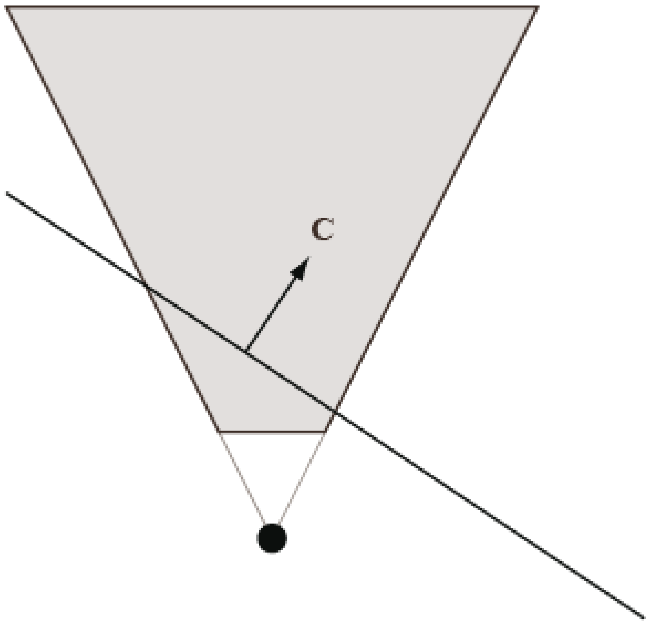
Oblique near plane trick

- Hardware clipping plane?
  - May not even be supported
  - Requires shader modification
  - Could be slower

Oblique near plane trick

- Extra clipping plane almost always redundant with near plane

- Don’t need to clip to both

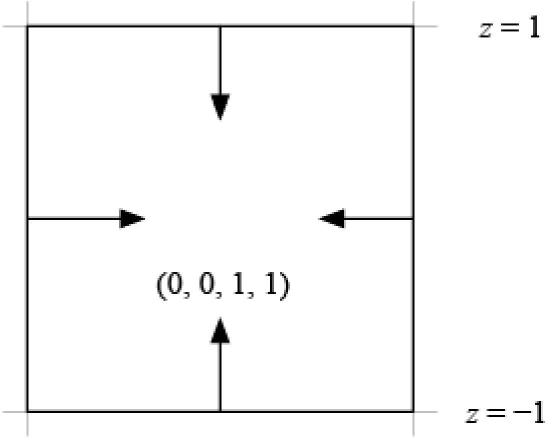


Oblique near plane trick

- Possible to modify projection matrix
- Move near plane to arbitrary location
- No extra clipping plane, no redundancy

Oblique near plane trick

- In normalized device coordinates (NDC), near plane has coordinates (0,0,1,1)



Oblique near plane trick

- Planes (row antivectors) are transformed from NDC to camera space by right multiplication by the projection matrix
- So the plane (0, 0, 1, 1) becomes $\mathbf{M}_3 + \mathbf{M}_4$, where $\mathbf{M}_i$ is the i-th row of the projection matrix

Oblique near plane trick

- $\mathbf{M}_4$ must remain $(0, 0, -1, 0)$ so that perspective correction still works right
- Let $\mathbf{C} = (C_x, C_y, C_z, C_w)$ be the camera-space plane that we want to clip against
  - Assume $C_w < 0$, camera on negative side
- We must have $\mathbf{C} = \mathbf{M}_3 + (0, 0, -1, 0)$

Oblique near plane trick

- $\mathbf{M}_3 = \mathbf{C} - \mathbf{M}_4 = (C_x, C_y, C_z + 1, C_w)$

$$\mathbf{M} = \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & e/a & 0 & 0 \\ C_x & C_y & C_z + 1 & C_w \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

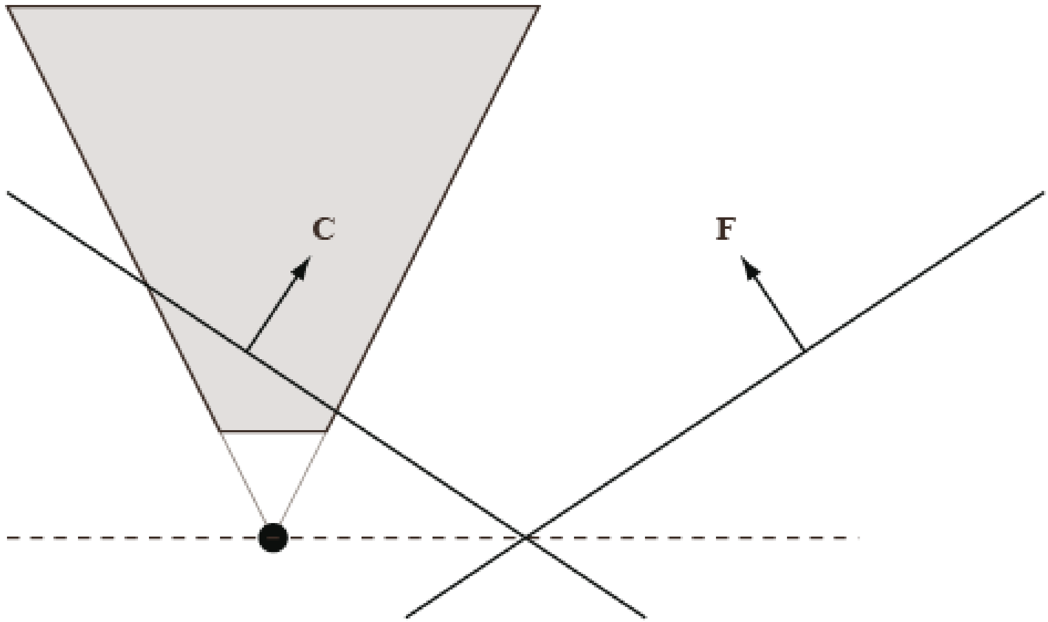
- This matrix maps points on the plane **C** to the plane $z = -1$ in NDC

Oblique near plane trick

- But what happens to the far plane?
- $\mathbf{F} = \mathbf{M}_4 - \mathbf{M}_3 = 2\mathbf{M}_4 - \mathbf{C}$

- Near plane and (negative) far plane differ only in the z coordinate
- Thus, they must coincide where they intersect the $z = 0$ plane

Oblique near plane trick



Oblique near plane trick

- Far plane is a complete mess
- Depths in NDC no longer represent distance from camera plane, but correspond to some skewed direction between near and far planes
- We can minimize the effect, and in practice it's not so bad

Oblique near plane trick

- We still have a free parameter: the clipping plane **C** can be scaled
- Scaling **C** has the effect of changing the orientation of the far plane **F**
- We want to make the new view frustum as small as possible while still including the conventional view frustum

Oblique near plane trick

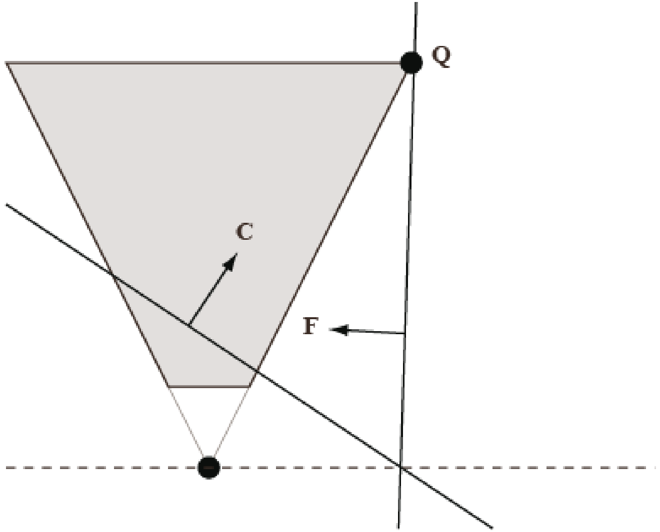
- Let $\mathbf{F} = 2\mathbf{M}_4 - a\mathbf{C}$
- Choose the point **Q** which lies furthest opposite the near plane in NDC:

$$\mathbf{Q} = \mathbf{M}^{-1} \cdot \left(\operatorname{sgn}\left(C_x\right), \operatorname{sgn}\left(C_y\right), 1, 1\right)$$

- Solve for *a* such that **Q** lies in plane **F**: $a = \dfrac{\mathbf{M}_4 \wedge \mathbf{Q}}{\mathbf{C} \wedge \mathbf{Q}}$

Oblique near plane trick

- Near plane doesn't move, but far plane becomes optimal



Oblique near plane trick

- Works for any perspective projection matrix
  - Even with infinite far depth

- More analysis available in "Oblique Depth Projection and View Frustum Clipping", Journal of Game Development, Vol. 1, No. 2.

Oblique near plane trick

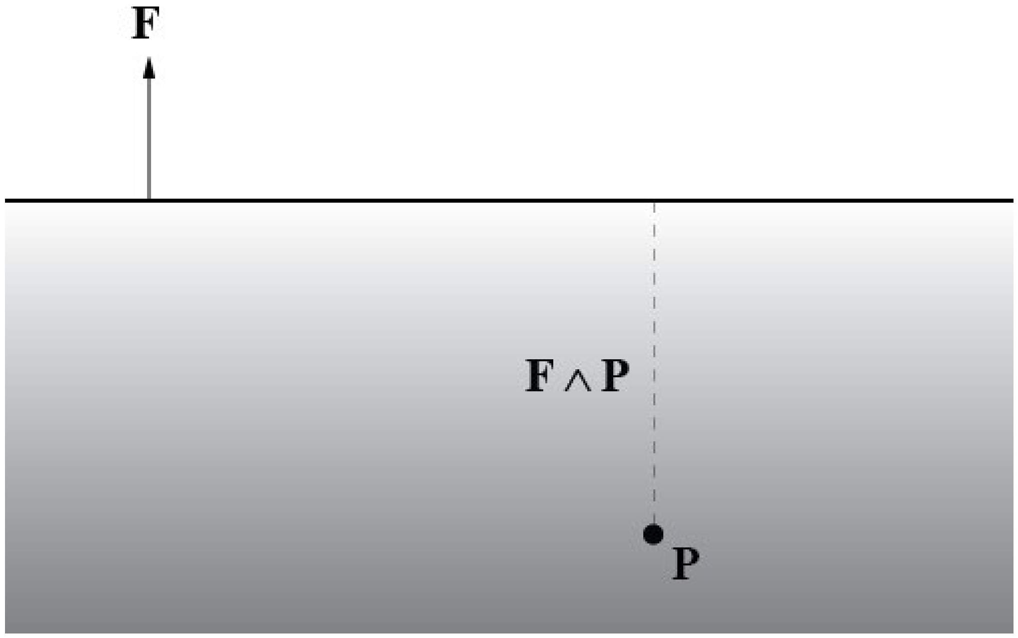
- Strategy:

Get the big picture

Fog bank occlusion

- Consider fog bank bounded by plane
- Linear density gradient with increasing depth
  - Perpendicular to plane
  - Zero density at plane

Fog bank occlusion



Fog bank occlusion

Fog bank occlusion

- For a given camera position, we want to cull objects that are completely fogged

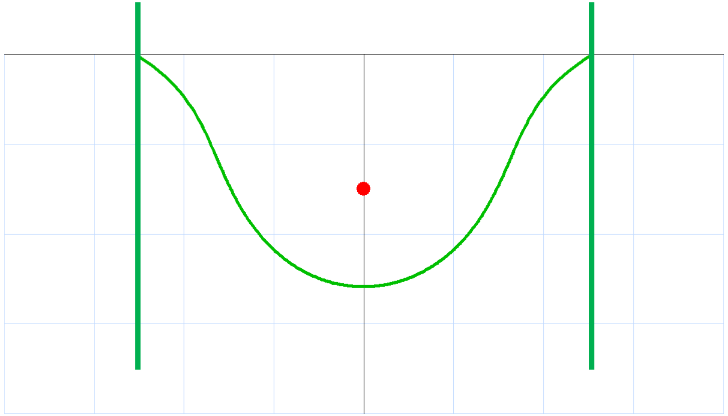
- Surface beyond which objects are completely fogged is interesting...

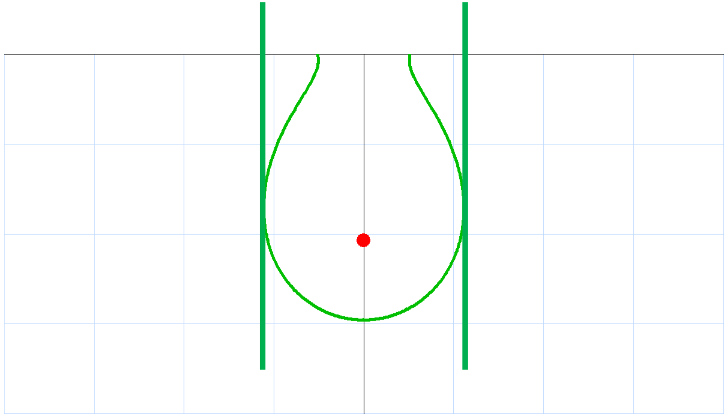
Fog bank occlusion

Fog bank occlusion

- Culling against that curve is impractical
- Instead, calculate its maximum extent parallel to the fog plane
- Then cull against plane perpendicular to fog plane and camera view direction at that distance

Fog bank occlusion





Fog bank occlusion

- **F** = fog plane, normal outward
- **C** = camera position
- **P** = point being shaded
- **V** = **C** – **P**
- *a* = linear density coefficient

Fog bank occlusion

- Density as function of depth:

$$\rho(\mathbf{P}) = -a(\mathbf{F} \wedge \mathbf{P})$$

- Log light fraction $g(\mathbf{P})$ for given **C** and **P**:†

$$g(\mathbf{P}) = -a\|\mathbf{V}\|\frac{\mathbf{F} \wedge \mathbf{P} + \mathbf{F} \wedge \mathbf{C}}{2}$$

†See "Unified Distance Formulas for Halfspace Fog", *Journal of Graphics Tools*, Vol. 12, No. 2 (2007).

Fog bank occlusion

- Set $g(\mathbf{P})$ to log of small enough fraction to be considered fully fogged
- For example: $g(\mathbf{P}) = -\log(1/256)$
- This is constant

Fog bank occlusion

- For given **C**, we need to find **P** with the maximum horizontal distance *d* from **C** that also satisfies

$$g(\mathbf{P}) = -a\|\mathbf{V}\|\frac{\mathbf{F}\wedge\mathbf{P}+\mathbf{F}\wedge\mathbf{C}}{2}$$

Fog bank occlusion

- So express *d* as a function of **P** and find the zeros of the derivative, right?
- Turns out to be a huge mess
- Not clear that good solution exists

Fog bank occlusion

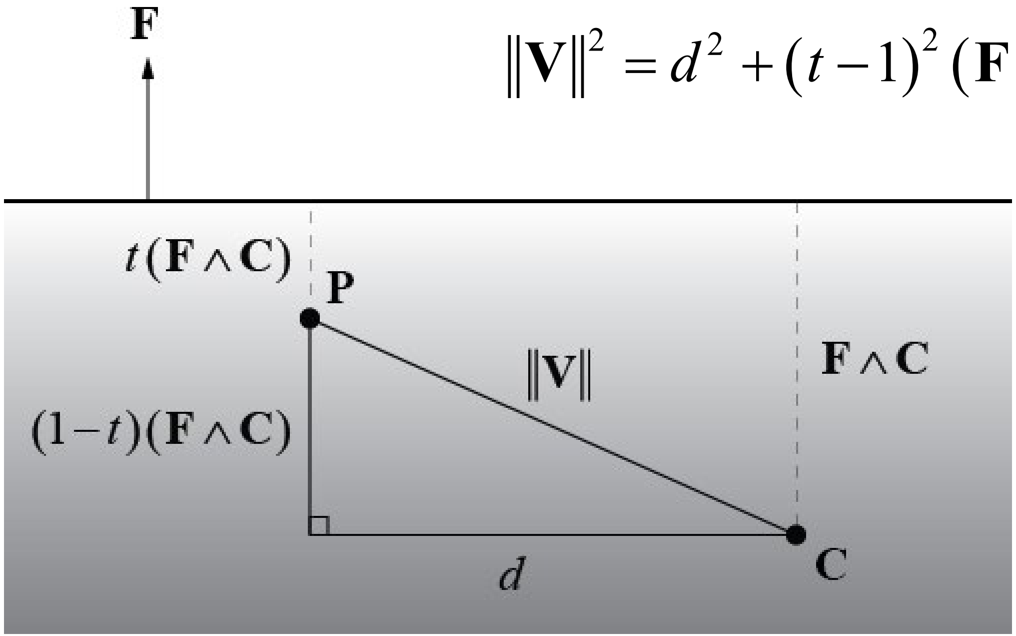
- Insight: instead of using independent variable **P**, express **F** ∧ **P** as a fraction of **F** ∧ **C**

$$\mathbf{F} \wedge \mathbf{P} = t(\mathbf{F} \wedge \mathbf{C})$$

$$g(\mathbf{P}) = -a\|\mathbf{V}\|\frac{(t+1)(\mathbf{F} \wedge \mathbf{C})}{2}$$

Fog bank occlusion

$$\|\mathbf{V}\|^2 = d^2 + (t-1)^2 (\mathbf{F} \wedge \mathbf{C})^2$$



Fog bank occlusion

- Can now write $g(\mathbf{P})$ as follows

$$g(\mathbf{P}) = -\frac{a}{2}(t+1)(\mathbf{F} \wedge \mathbf{C})\sqrt{d^2 + (t-1)^2 (\mathbf{F} \wedge \mathbf{C})^2}$$

- And solve for d^2:

$$d^2 = \frac{m^2}{(t+1)^2 (\mathbf{F} \wedge \mathbf{C})^2} - (t-1)^2 (\mathbf{F} \wedge \mathbf{C})^2 \qquad m = \frac{2g(\mathbf{P})}{a}$$

Fog bank occlusion

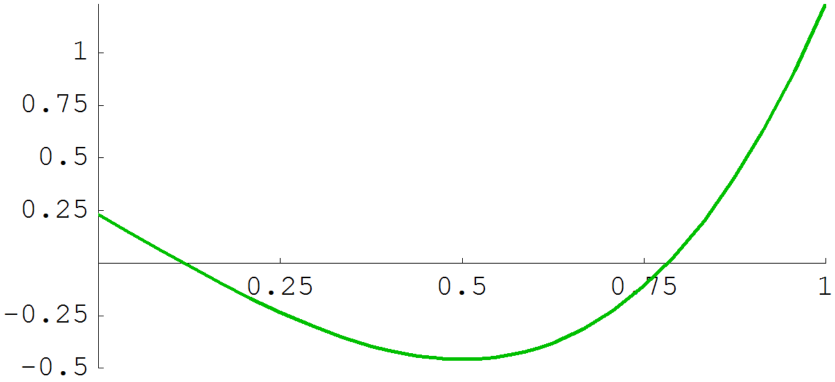
- Take derivative, set to zero, simplify:

$$t^4 + 2t^3 - 2t + k - 1 = 0 \qquad k = \frac{m^2}{(\mathbf{F} \wedge \mathbf{C})^4}$$

- Now need to solve quartic polynomial

Fog bank occlusion

- Know what your functions look like!



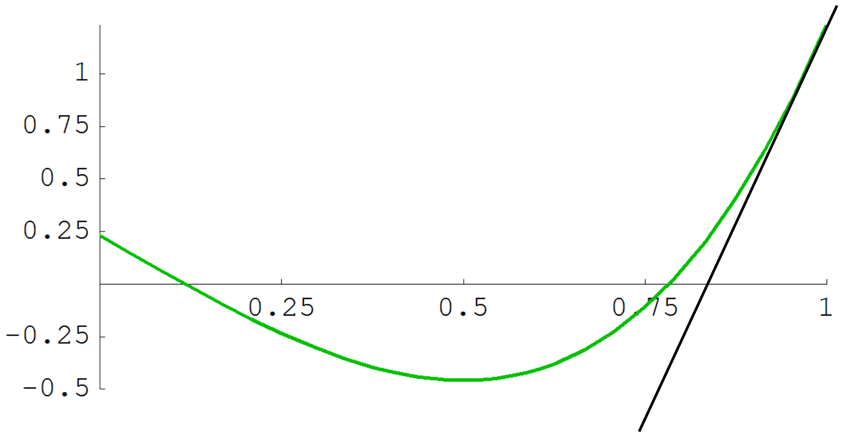
- Always k at $t = 1$, local min at $t = 1/2$

Fog bank occlusion

- If function is negative at $t = 1/2$, then solution exists
  - Happens exactly when $k < 27/16$
- Tempting to calculate with closed-form solution to quartic
- But almost always better to use Newton's method, especially in this case

Fog bank occlusion

- Newton's method: $$t_{i+1} = t_i - \frac{f(t)}{f'(t)}$$



Fog bank occlusion

- In our case,

$$f'(t) = 4t^3 + 6t^2 - 2$$

- Start with $t_0 = 1$:

$$f(1) = k$$
$$f'(1) = 8$$

Fog bank occlusion

- Calculate first iteration explicitly:

$$t_1 = 1 - \frac{k}{8}$$

- Newton's method converges very quickly with 1–2 more iterations

Fog bank occlusion

- Plug t back into function for d^2 to get culling distance

$$d^2 = \frac{m^2}{(t+1)^2(\mathbf{F} \wedge \mathbf{C})^2} - (t-1)^2(\mathbf{F} \wedge \mathbf{C})^2$$

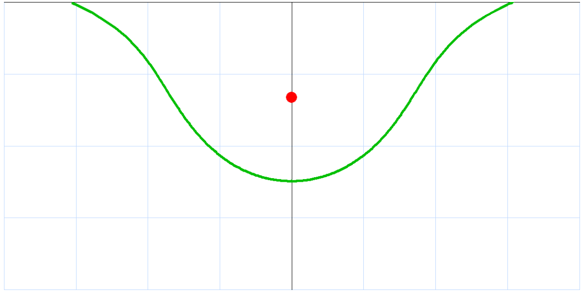
- If $d^2 > 0$ when $t = 0$, possible larger culling distance

Fog bank occlusion

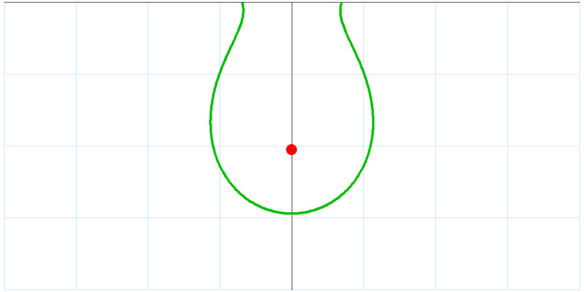
- Solution exists at $t = 0$ when $k > 1$
- Solution exists deeper when $k < 27/16$
- Take the larger distance if both exist

Fog bank occlusion

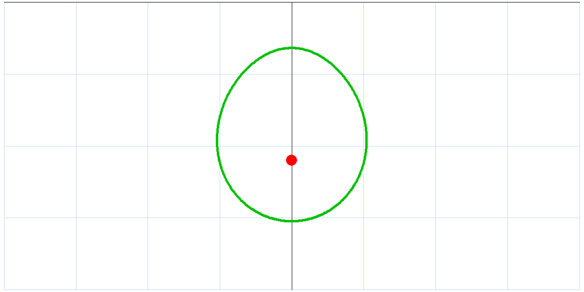
$k > \frac{27}{16}$



$k \in \left[1, \frac{27}{16}\right]$



$k < 1$



Fog bank occlusion

- Strategy:

  Eliminate variables
  Know what functions look like
  Embrace Newton’s method

Contact

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Supplemental slides

- Bézier animation curves
- Floor and ceiling functions
- Cross product trick
- Bit manipulation tricks

Bézier animation curves

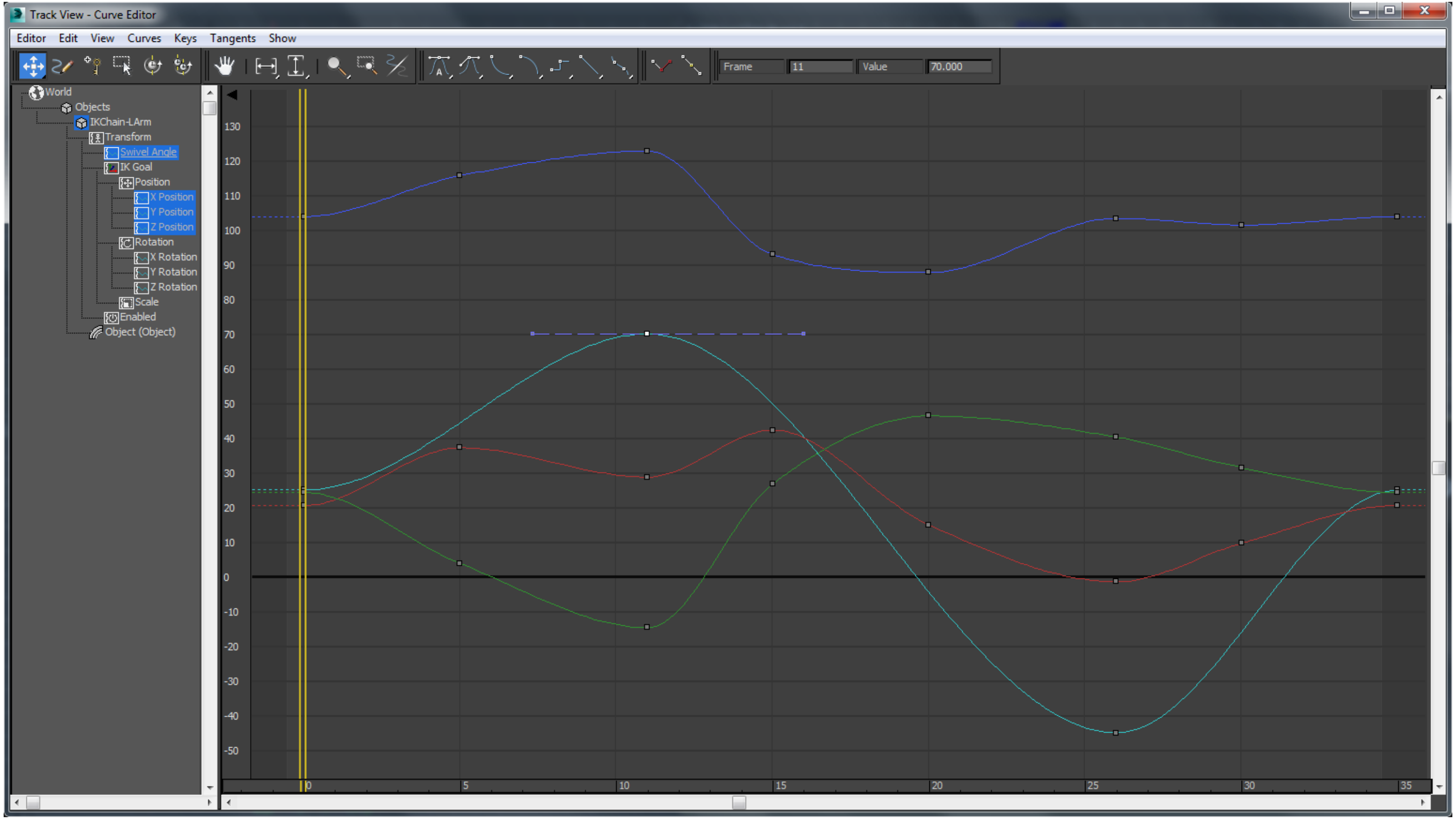
- Another note about Newton's method
- Cubic 2D Bézier curves often used to animate some component of an object's transform
- Position *x*, *y*, *z* or rotation angle, for example

Bézier animation curves

- 2D coordinates on curve are time *t* and some scalar value *v*
- *t* is *not* the parameter along the curve
- Big source of confusion in data exchange

Bézier animation curves

- Control points specified in (t, v) space
- Time coordinates increase monotonically

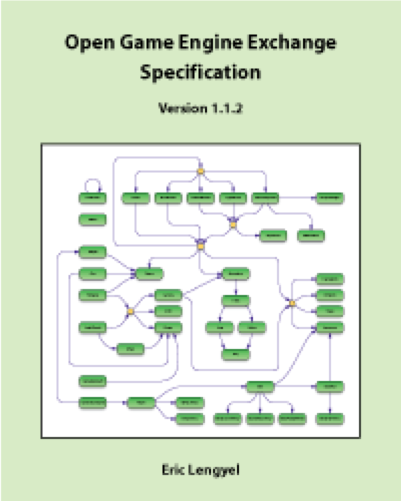

Track View - Curve Editor
Editor
Edit
View
Curves
Keys
Tangents
Show
Frame
11
Value
70.000
World
Objects
IKChain-LArm
Transform
Swivel Angle
IK Goal
Position
X Position
Y Position
Z Position
Rotation
X Rotation
Y Rotation
Z Rotation
Scale
Enabled
Object (Object)
130
120
110
100
90
80
70
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
0
5
10
15
20
25
30
35

Bézier animation curves

- To evaluate the value of a curve **P**(s) at a given time t, it's necessary to find the parameter s along the curve for which $P_t(s) = t$
- Requires solving a cubic polynomial
- Newton's method perfect for this case

Bézier animation curves

- For details, see `Track` structure in OpenGEX Specification
- opengex.org



Floor and ceiling

- Not all CPUs have floating-point floor/ceil/trunc/round instructions
- Need to implement with ordinary math
- Needs to be fast, no FP/int conversions

Floor function

- 32-bit float has 23 bits in mantissa
- Thus, any value greater than or equal to 2^{23} is necessarily an integer
  - No bits left for any fractional part

Floor function

- Trick is to add and subtract 2^{23}
- The addition causes all fraction bits to be shifted out the right end
- The subtraction shifts zeros back into the space previously occupied by the fraction

Floor function

- When we add 2^{23}, the original number is rounded to the nearest integer + 2^{23}
- If result is greater than original number, then simply subtract one to get floor

Floor function

- What about negative numbers?
- Use the same trick, but subtract 2^{23} first, and then add it back
- Can combine for all possible inputs

Floor function

```
__m128 floor(__m128 x)
{
    __m128 one = {0x3F800000};
    __m128 two23 = {0x4B000000};
    __m128 f = _mm_sub_ps(_mm_add_ps(f, two23), two23);
    f = _mm_add_ps(_mm_sub_ps(x, two23), two23);
    f = _mm_sub_ps(f, _mm_and_ps(one, _mm_cmplt_ps(x, f)));
    return (f);
}
```

Floor function

- But wait, this fails for some very large inputs (bigger than 2^{23})
- All of these inputs are already integers!
  - They must be if they're bigger than 2^{23}

Floor function

- So just return the input if it’s > 2^{23}

```
__m128 sgn = {0x80000000};
__m128 msk = _mm_cmplt_ps(two23, _mm_andnot_ps(sgn, x));
f = _mm_or_ps(_mm_andnot_ps(msk, f), _mm_and_ps(msk, x));
```

Ceiling function

- Instead of subtracting one if result is greater than input, add one if result is less than input

```
f = _mm_add_ps(f, _mm_and_ps(one, _mm_cmplt_ps(f, x)));
```

Floor and ceiling

- Strategy:

  Reduce problem domain

Cross product trick

- Cross product $V \times W$ given by:

$$V.yzx * W.zxy - W.zxy * V.yzx$$

- Two mults, one sub, **four** shuffles

Cross product trick

- Can do this instead:

$$(V * W.yzx - V.yzx * W).yzx$$

- Two mults, one sub, **three** shuffles
  - And same shuffle each time

Bit manipulation tricks

- Range checks
- Non-branching calculations
- Logic formulas

Integer range checks

- Integer range checks can always be done with a single comparison:

```
(unsigned) (x - min) <= (unsigned) (max - min)
```

Non-branching calculations

- Using logic tricks to avoid branches in integer calculations
- Many involve using sign bit in clever way
- Also useful to know `-x == ~x + 1`

Non-branching calculations

- Helps scheduling, increases ILP
- Reduces pollution in branch history table
- But can obfuscate code
  - Use where performance is very important
  - Don't bother elsewhere

Clever uses of sign bit

- `if (a < 0) ++x;`
- Replace with:
- `x -= a >> 31;      // 32-bit ints`

Right-shifting negative integers

- Shifting n-bit int right by $n - 1$ bits:
  - All zeros for positive ints (or zero)
  - All ones for negative ints
- C++ standard says `a >> 31` is "implementation-defined" if `a` is negative

Right-shifting negative integers

- Any sensible compiler generates instruction that replicates sign bit
- To avoid issue in this case, could also use:
- `x += (uint32) a >> 31`

Predicates for 32-bit signed ints

- `(x == 0)` `lzcnt(x) >> 5`
- `(x != 0)` `(lzcnt(x) >> 5) ^ 1`
- `(x < 0)` `(uint32) x >> 31`
- `(x > 0)` `(uint32) -x >> 31`
- `(x == y)` `lzcnt(x - y) >> 5`
- `(x != y)` `(uint32) ((x - y) | (y - x)) >> 31`

- lzcnt() is leading zero count

Absolute value

- `y = x >> 31`
- `abs(x) = (x ^ y) - y`

- Because `-x = ~x + 1`

  `= x ^ 0xFFFFFFFF - 0xFFFFFFFF`

Conditional negation

- Same trick can be used to negate for any bool condition:

- `if (condition) x = -x;`

- `x = (x ^ -condition) + condition`

Logic Formulas

Formula	Operation / Effect	Notes
`x & (x - 1)`	Clear lowest 1 bit.	If result is 0, then x is 2^n.
`x | (x + 1)`	Set lowest 0 bit.	
`x | (x - 1)`	Set all bits to right of lowest 1 bit.	
`x & (x + 1)`	Clear all bits to right of lowest 0 bit.	If result is 0, then x is $2^n - 1$.
`x & -x`	Extract lowest 1 bit.	
`~x & (x + 1)`	Extract lowest 0 bit (as a 1 bit).	
`~x | (x - 1)`	Create mask for bits other than lowest 1 bit.	
`x | ~(x + 1)`	Create mask for bits other than lowest 0 bit.	
`x | -x`	Create mask for bits left of lowest 1 bit, inclusive.	
`x ^ -x`	Create mask for bits left of lowest 1 bit, exclusive.	
`~x | (x + 1)`	Create mask for bits left of lowest 0 bit, inclusive.	
`~x ^ (x + 1)`	Create mask for bits left of lowest 0 bit, exclusive.	Also x ≡ (x + 1).
`x ^ (x - 1)`	Create mask for bits right of lowest 1 bit, inclusive.	0 becomes −1.
`~x & (x - 1)`	Create mask for bits right of lowest 1 bit, exclusive.	0 becomes −1.
`x ^ (x + 1)`	Create mask for bits right of lowest 0 bit, inclusive.	remains −1.
`x & (~x - 1)`	Create mask for bits right of lowest 0 bit, exclusive.	remains −1.

This table from “Bit Hacks for Games”, *Game Engine Gems 2*, A K Peters, 2011.

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