Fundamentals of Grassmann Algebra

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Math used in 3D programming

- Dot / cross products, scalar triple product
- Planes as 4D vectors
- Homogeneous coordinates
- Plücker coordinates for 3D lines
- Transforming normal vectors and planes with the inverse transpose of a matrix

Math used in 3D programming

- These concepts often used without a complete understanding of the big picture
  - Can be used in a way that is not natural
  - Different pieces used separately without knowledge of the connection among them

There is a bigger picture

- All of these arise as part of a single mathematical system
  - Understanding the big picture provides deep insights into seemingly unusual properties
  - Knowledge of the relationships among these concepts makes better 3D programmers

History

- Hamilton, 1843
  - Discovered quaternion product
  - Applied to 3D rotations
  - Not part of Grassmann algebra

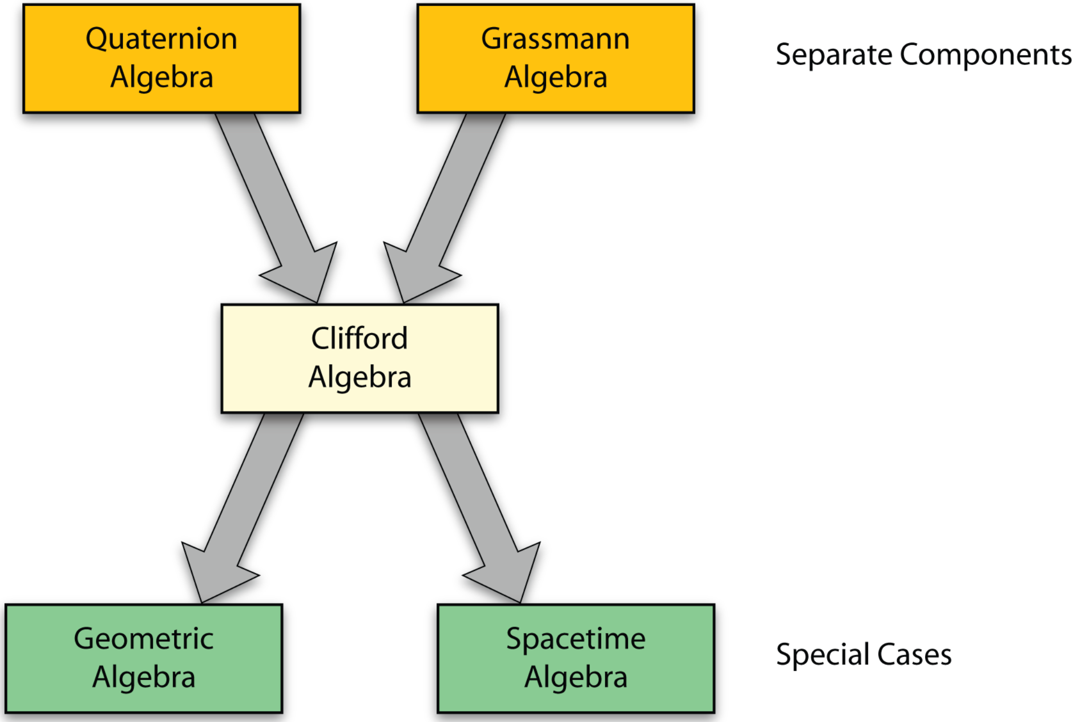
History

- Grassmann, 1844
  - Formulated progressive and regressive products
  - Understood geometric meaning
  - Published "Algebra of Extension"

History

- Clifford, 1878
  - Unified Hamilton's and Grassmann's work
  - Basis for modern geometric algebra and various algebras used in physics

History


Quaternion Algebra
Grassmann Algebra
Separate Components
Clifford Algebra
Geometric Algebra
Spacetime Algebra
Special Cases

Outline

- Grassmann algebra in 3-4 dimensions
  - Wedge product, bivectors, trivectors...
  - Transformations
  - Homogeneous model
  - Geometric computation
  - Programming considerations

The wedge product

- Also known as:
  - The progressive product
  - The exterior product
- Gets name from symbol:

$$\mathbf{a} \wedge \mathbf{b}$$

- Read “**a** *wedge* **b**”

The wedge product

- Operates on scalars, vectors, and more
  - Ordinary multiplication for scalars *s* and *t*:

$$s \wedge t = t \wedge s = st$$

$$s \wedge \mathbf{v} = \mathbf{v} \wedge s = s\mathbf{v}$$

  - The square of a vector **v** is always zero:

$$\mathbf{v} \wedge \mathbf{v} = 0$$

Wedge product anticommutativity

- Zero square implies vectors anticommute

$$(\mathbf{a}+\mathbf{b})\wedge(\mathbf{a}+\mathbf{b})=0$$

$$\mathbf{a}\wedge\mathbf{a}+\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{a}+\mathbf{b}\wedge\mathbf{b}=0$$

$$\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{a}=0$$

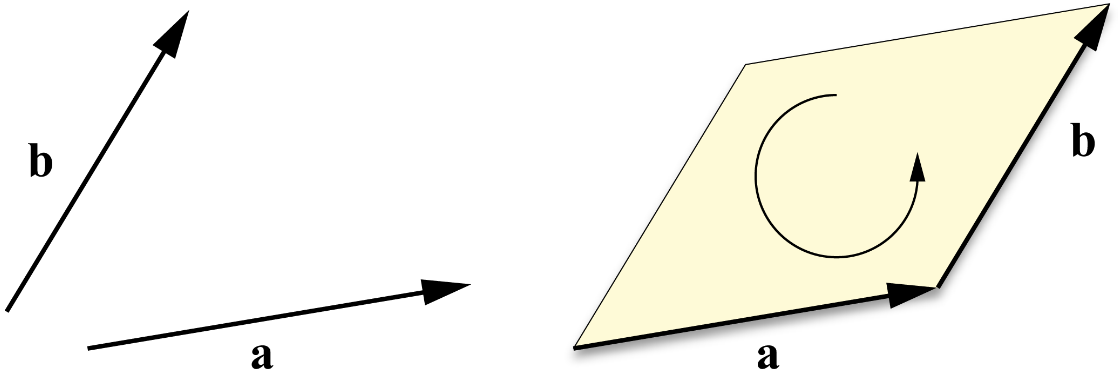
$$\mathbf{a}\wedge\mathbf{b}=-\mathbf{b}\wedge\mathbf{a}$$

Bivectors

- Wedge product between two vectors produces a “bivector”
  - A new mathematical entity
  - Distinct from a scalar or vector
  - Represents an oriented 2D area
    - Whereas a vector represents an oriented 1D direction
    - Scalars are zero-dimensional values

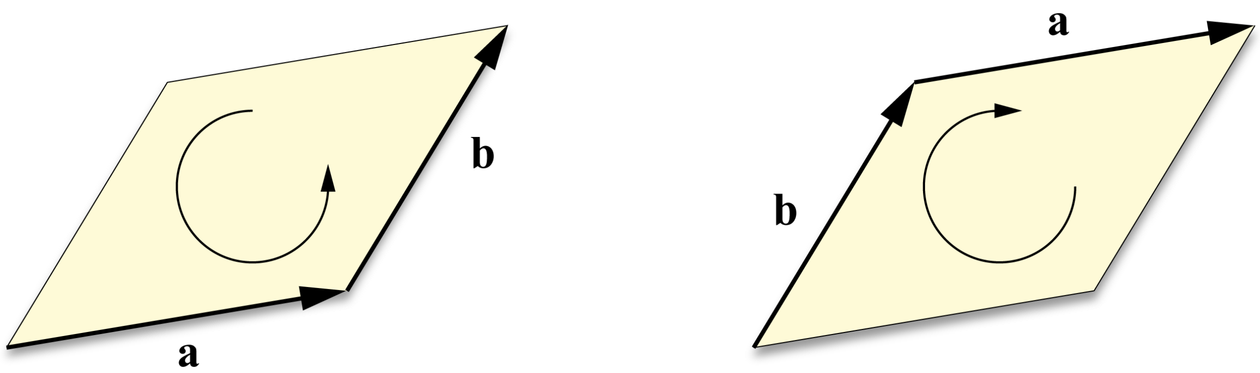
Bivectors

- Bivector is two directions and magnitude



Bivectors

- Order of multiplication matters



$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

Bivectors in 3D

- Start with 3 orthonormal basis vectors:

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$$

- Then a 3D vector **a** can be expressed as

$$a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$$

Bivectors in 3D

$$\mathbf{a} \wedge \mathbf{b} = (a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \wedge (b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3)$$

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} = {} & a_1b_2(\mathbf{e}_1 \wedge \mathbf{e}_2) + a_1b_3(\mathbf{e}_1 \wedge \mathbf{e}_3) + a_2b_1(\mathbf{e}_2 \wedge \mathbf{e}_1) \\ & + a_2b_3(\mathbf{e}_2 \wedge \mathbf{e}_3) + a_3b_1(\mathbf{e}_3 \wedge \mathbf{e}_1) + a_3b_2(\mathbf{e}_3 \wedge \mathbf{e}_2)\end{aligned}$$

$$\begin{aligned}\mathbf{a} \wedge \mathbf{b} = {} & (a_2b_3 - a_3b_2)(\mathbf{e}_2 \wedge \mathbf{e}_3) + (a_3b_1 - a_1b_3)(\mathbf{e}_3 \wedge \mathbf{e}_1) \\ & + (a_1b_2 - a_2b_1)(\mathbf{e}_1 \wedge \mathbf{e}_2)\end{aligned}$$

Bivectors in 3D

- The result of the wedge product has three components on the basis

$$\mathbf{e}_2 \wedge \mathbf{e}_3, \quad \mathbf{e}_3 \wedge \mathbf{e}_1, \quad \mathbf{e}_1 \wedge \mathbf{e}_2$$

- Written in order of which basis *vector* is missing from the basis *bivector*

Bivectors in 3D

- Do the components look familiar?

$$\mathbf{a} \wedge \mathbf{b} = (a_2 b_3 - a_3 b_2)(\mathbf{e}_2 \wedge \mathbf{e}_3) + (a_3 b_1 - a_1 b_3)(\mathbf{e}_3 \wedge \mathbf{e}_1) + (a_1 b_2 - a_2 b_1)(\mathbf{e}_1 \wedge \mathbf{e}_2)$$

- These are identical to the components produced by the cross product **a** × **b**

Shorthand notation

$$\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$$

$$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$$

$$\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$$

$$\mathbf{e}_{123} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$$

Bivectors in 3D

$$\mathbf{a} \wedge \mathbf{b} = (a_2 b_3 - a_3 b_2)\mathbf{e}_{23} + (a_3 b_1 - a_1 b_3)\mathbf{e}_{31} + (a_1 b_2 - a_2 b_1)\mathbf{e}_{12}$$

Comparison with cross product

- The cross product is not associative:

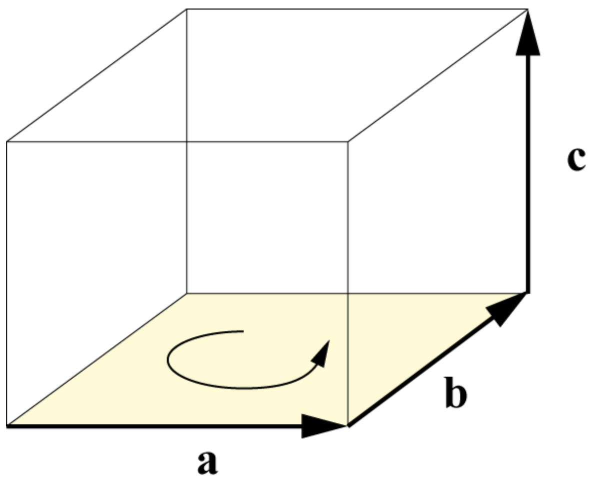
$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\neq\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$$

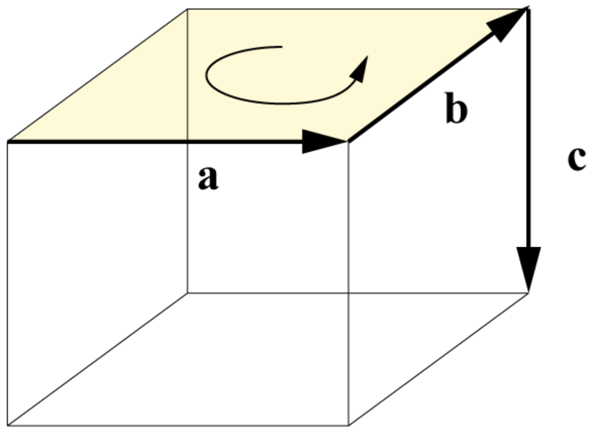
- The cross product is only defined in 3D
- The wedge product is associative, and it's defined in all dimensions

Trivectors

- Wedge product among three vectors produces a "trivector"
  - Another new mathematical entity
  - Distinct from scalars, vectors, and bivectors
  - Represents a 3D oriented volume

Trivectors





$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$$

Trivectors in 3D

- A 3D trivector has one component:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} =$$
$$(a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1) \cdot$$
$$(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$$

- The magnitude is $\det([\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}])$

Trivectors in 3D

- 3D trivector also called *pseudoscalar* or antiscalar
  - Only one component, so looks like a scalar
  - Flips sign under reflection

Scalar Triple Product

- The product

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$$

produces the same magnitude as

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

but also extends to higher dimensions

Grading

- The *grade* of an entity is the number of vectors wedged together to make it
  - Scalars have grade 0
  - Vectors have grade 1
  - Bivectors have grade 2
  - Trivectors have grade 3
  - Etc.

3D multivector algebra

- 1 scalar element
- 3 vector elements
- 3 bivector elements
- 1 trivector element
- No higher-grade elements
- Total of 8 *multivector* basis elements

Multivectors in general dimension

- In *n* dimensions, the number of basis *k*-vector elements is

$$\binom{n}{k}$$

- This produces a nice symmetry
- Total number of basis elements always 2^n

Multivectors in general dimension

Dimension	Graded elements
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Four dimensions

- Four basis vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
- Number of basis bivectors is

$$\binom{4}{2} = 6$$

- There are 4 basis trivectors

Vector / bivector confusion

- In 3D, vectors have three components
- In 3D, bivectors have three components
- Thus, vectors and bivectors *look like* the same thing!
- This is a big reason why knowledge of the difference is not widespread

Cross product peculiarities

- Physicists noticed a long time ago that the cross product produces a *different* kind of vector
  - They call it an “axial vector”, “pseudovector”, “covector”, or “covariant vector”
  - It transforms differently than ordinary “polar vectors” or “contravariant vectors”

Cross product transform

- Simplest example is a reflection:

$$\mathbf{M} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

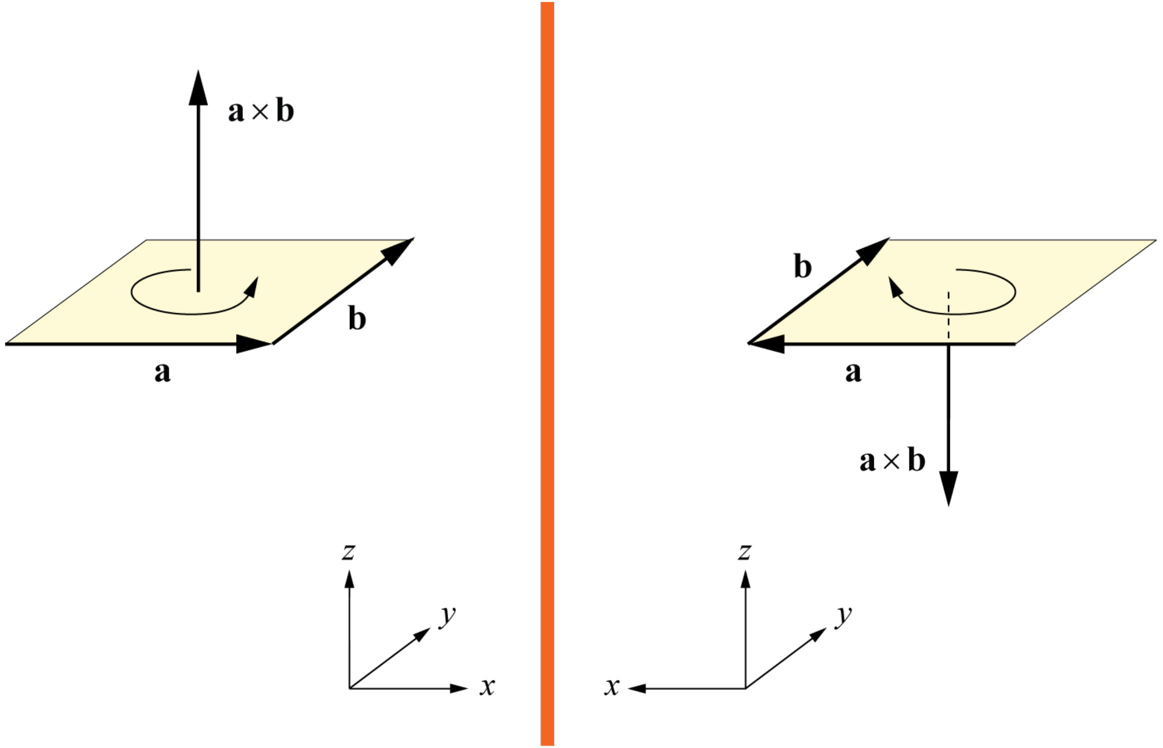
Cross product transform

$$(1,0,0)\times(0,1,0)=(0,0,1)$$

$$\mathbf{M}(1,0,0)\times\mathbf{M}(0,1,0)$$
$$=(-1,0,0)\times(0,1,0)=(0,0,-1)$$

- Not the same as $\mathbf{M}(0,0,1)=(0,0,1)$

Cross product transform



Cross product transform

- In general, for 3 x 3 matrix **M**,

$$\mathbf{M}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1\mathbf{M}_1 + a_2\mathbf{M}_2 + a_3\mathbf{M}_3$$

$$\mathbf{Ma} \times \mathbf{Mb} =$$

$$(a_1\mathbf{M}_1 + a_2\mathbf{M}_2 + a_3\mathbf{M}_3) \times (b_1\mathbf{M}_1 + b_2\mathbf{M}_2 + b_3\mathbf{M}_3)$$

Cross product transform

$$\mathbf{Ma} \times \mathbf{Mb} =$$
$$(a_2 b_3 - a_3 b_2)(\mathbf{M}_2 \times \mathbf{M}_3)$$
$$+(a_3 b_1 - a_1 b_3)(\mathbf{M}_3 \times \mathbf{M}_1)$$
$$+(a_1 b_2 - a_2 b_1)(\mathbf{M}_1 \times \mathbf{M}_2)$$

Products of matrix columns

$$(\mathbf{M}_2 \times \mathbf{M}_3) \cdot \mathbf{M}_1 = \det \mathbf{M}$$

$$(\mathbf{M}_3 \times \mathbf{M}_1) \cdot \mathbf{M}_2 = \det \mathbf{M}$$

$$(\mathbf{M}_1 \times \mathbf{M}_2) \cdot \mathbf{M}_3 = \det \mathbf{M}$$

- Other dot products are zero

Matrix inversion

- Cross products as rows of matrix:

$$\begin{bmatrix} \mathbf{M}_2 \times \mathbf{M}_3 \\ \mathbf{M}_3 \times \mathbf{M}_1 \\ \mathbf{M}_1 \times \mathbf{M}_2 \end{bmatrix} \mathbf{M} = \begin{bmatrix} \det \mathbf{M} & 0 & 0 \\ 0 & \det \mathbf{M} & 0 \\ 0 & 0 & \det \mathbf{M} \end{bmatrix}$$

Cross product transform

- Transforming the cross product requires the inverse matrix:

$$\begin{bmatrix} \mathbf{M}_2 \times \mathbf{M}_3 \\ \mathbf{M}_3 \times \mathbf{M}_1 \\ \mathbf{M}_1 \times \mathbf{M}_2 \end{bmatrix} = (\det \mathbf{M})\mathbf{M}^{-1}$$

Cross product transform

- Transpose the inverse to get right result:

$$(\det \mathbf{M})\mathbf{M}^{-T}\begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} =$$

$$(a_2b_3 - a_3b_2)(\mathbf{M}_2 \times \mathbf{M}_3) + (a_3b_1 - a_1b_3)(\mathbf{M}_3 \times \mathbf{M}_1)$$

$$+(a_1b_2 - a_2b_1)(\mathbf{M}_1 \times \mathbf{M}_2)$$

Cross product transform

- Transformation formula:

$$\mathbf{Ma} \times \mathbf{Mb} = (\det \mathbf{M})\,\mathbf{M}^{-T}(\mathbf{a} \times \mathbf{b})$$

- Result of cross product must be transformed by inverse transpose times determinant

Cross product transform

- If **M** is orthogonal, then inverse transpose is the same as **M**
- If the determinant is positive, then it can be left out if you don't care about length
- Determinant times inverse transpose is called *adjugate transpose*

Cross product transform

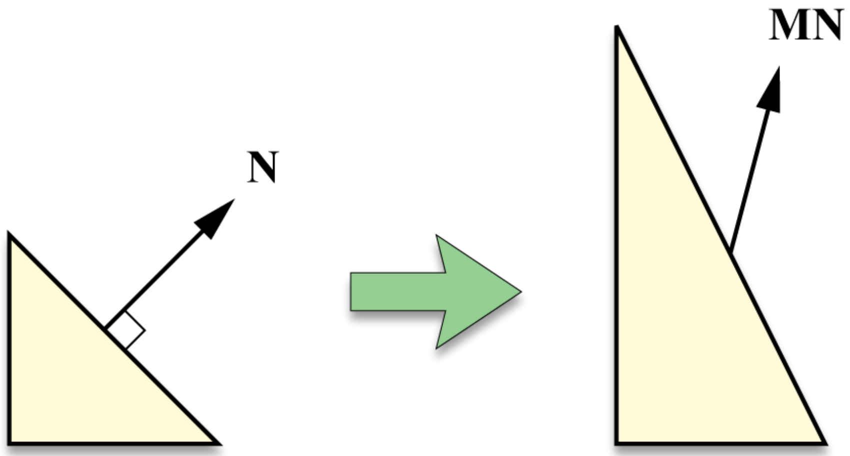
- What’s really going on here?

- When we take a cross product, we are really creating a bivector
- Bivectors are not vectors, and they don’t behave like vectors

Normal “vectors”

- A triangle normal is created by taking the cross product between two tangent vectors
- A normal is a bivector and transforms as such

Normal "vector" transformation



Classical derivation

- Standard proof for inverse transpose for transforming normals:
  - Preserve zero dot product with tangent
  - Misses extra factor of det **M**

$$\mathbf{N} \cdot \mathbf{T} = 0$$

$$\mathbf{UN} \cdot \mathbf{MT} = 0$$

$$\mathbf{N}^T \mathbf{U}^T \mathbf{MT} = 0$$

$$\mathbf{U}^T = \mathbf{M}^{-1}$$

$$\mathbf{U} = \mathbf{M}^{-T}$$

Matrix inverses

- In general, the *i*-th row of the inverse of **M** is 1/det **M** times the wedge product of all columns of **M** except column *i*.

Higher dimensions

- In *n* dimensions, the $(n-1)$-vectors have *n* components, just as 1-vectors do
- Each 1-vector basis element uses exactly one of the spatial directions $\mathbf{e}_1 \ldots \mathbf{e}_n$
- Each $(n-1)$-vector basis element uses all *except* one of the spatial directions $\mathbf{e}_1 \ldots \mathbf{e}_n$

Symmetry in three dimensions

- Vector basis and bivector ($n-1$) basis

$\mathbf{e}_1$	$\mathbf{e}_2 \wedge \mathbf{e}_3$
$\mathbf{e}_2$	$\mathbf{e}_3 \wedge \mathbf{e}_1$
$\mathbf{e}_3$	$\mathbf{e}_1 \wedge \mathbf{e}_2$

Symmetry in four dimensions

- Vector basis and trivector ($n-1$) basis

$$\mathbf{e}_1 \qquad \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$$

$$\mathbf{e}_2 \qquad \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_3$$

$$\mathbf{e}_3 \qquad \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$$

$$\mathbf{e}_4 \qquad \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2$$

Dual basis

- Use special notation for wedge product of all but one basis vector:

$$\overline{\mathbf{e}}_1 = \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$$

$$\overline{\mathbf{e}}_2 = \mathbf{e}_1 \wedge \mathbf{e}_4 \wedge \mathbf{e}_3$$

$$\overline{\mathbf{e}}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_4$$

$$\overline{\mathbf{e}}_4 = \mathbf{e}_1 \wedge \mathbf{e}_3 \wedge \mathbf{e}_2$$

Dual basis

- Instead of saying $(n-1)$-vector, we call these "antivectors"
- In n dimensions, antivector always means a quantity expressed on the basis with grade $n-1$

Vector / antivector product

- Wedge product between vector and antivector is the origin of the dot product

$$(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) \wedge (b_1\overline{\mathbf{e}}_1 + b_2\overline{\mathbf{e}}_2 + b_3\overline{\mathbf{e}}_3)$$

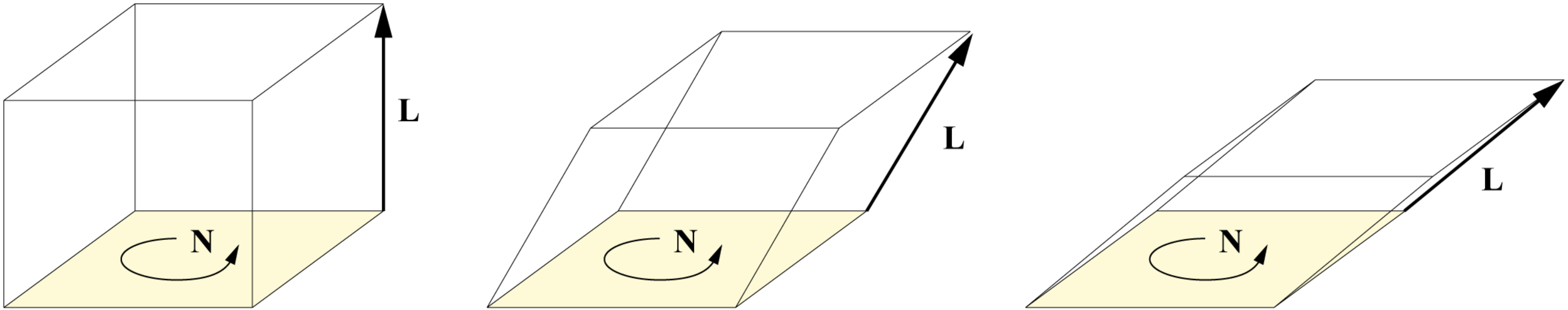
$$= (a_1b_1 + a_2b_2 + a_3b_3)(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3)$$

- They complement each other, and “fill in” the volume element

Vector / antivector product

- Many of the dot products you take are actually vector / antivector wedge products
- For instance, **N • L** in diffuse lighting
- **N** is an antivector
- Calculating volume of extruded bivector

Diffuse Lighting



The regressive product

- Grassmann realized there is another product symmetric to the wedge product
- Not well-known at all
  - Most books on geometric algebra leave it out completely
- Very important product, though!

The regressive product

- Operates on antivectors in a manner symmetric to how the wedge product operates on vectors
- Uses an upside-down wedge:

$$\overline{\mathbf{e}}_1 \vee \overline{\mathbf{e}}_2$$

- We call it the “antiwedge” product

The antiwedge product

- Has same properties as wedge product, but for antivectors
- Operates in complementary space on dual basis or "antibasis"

The antiwedge product

- Whereas the wedge product increases grade, the antiwedge product decreases it
- Suppose, in *n*-dimensional Grassmann algebra, **A** has grade *r* and **B** has grade *s*
- Then $\mathbf{A} \wedge \mathbf{B}$ has grade $r + s$
- And $\mathbf{A} \vee \mathbf{B}$ has grade

$$n - (n - r) - (n - s) = r + s - n$$

Antiwedge product in 3D

$$\overline{\mathbf{e}}_1 \vee \overline{\mathbf{e}}_2 = (\mathbf{e}_2 \wedge \mathbf{e}_3) \vee (\mathbf{e}_3 \wedge \mathbf{e}_1) = \mathbf{e}_3$$

$$\overline{\mathbf{e}}_2 \vee \overline{\mathbf{e}}_3 = (\mathbf{e}_3 \wedge \mathbf{e}_1) \vee (\mathbf{e}_1 \wedge \mathbf{e}_2) = \mathbf{e}_1$$

$$\overline{\mathbf{e}}_3 \vee \overline{\mathbf{e}}_1 = (\mathbf{e}_1 \wedge \mathbf{e}_2) \vee (\mathbf{e}_2 \wedge \mathbf{e}_3) = \mathbf{e}_2$$

Similar shorthand notation

$$\overline{\mathbf{e}}_{12} = \overline{\mathbf{e}}_1 \vee \overline{\mathbf{e}}_2$$

$$\overline{\mathbf{e}}_{23} = \overline{\mathbf{e}}_2 \vee \overline{\mathbf{e}}_3$$

$$\overline{\mathbf{e}}_{31} = \overline{\mathbf{e}}_3 \vee \overline{\mathbf{e}}_1$$

$$\overline{\mathbf{e}}_{123} = \overline{\mathbf{e}}_1 \vee \overline{\mathbf{e}}_2 \vee \overline{\mathbf{e}}_3$$

Join and meet

- Wedge product joins *vectors* together
  - Analogous to union
- Antiwedge product joins *antivectors*
  - Antivectors represent absence of geometry
  - Joining antivectors is like removing vectors
  - Analogous to intersection
  - Called a meet operation

Homogeneous coordinates

- Points have a 4D representation:

$$\mathbf{P} = (x, y, z, w)$$

- Conveniently allows affine transformation through 4 x 4 matrix
- Used throughout 3D graphics

Homogeneous points

- To project onto 3D space, find where 4D vector intersects subspace where $w = 1$

$$\mathbf{P} = (x, y, z, w)$$

$$\mathbf{P}_{\mathrm{3D}} = \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w} \right)$$

Homogeneous model

- With Grassmann algebra, homogeneous model can be extended to include 3D points, lines, and planes
- Wedge and antiwedge products naturally perform union and intersection operations among all of these

4D Grassmann Algebra

- Scalar unit
- Four vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$
- Six bivectors: $\mathbf{e}_{12}, \mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{41}, \mathbf{e}_{42}, \mathbf{e}_{43}$
- Four antivectors: $\overline{\mathbf{e}}_1, \overline{\mathbf{e}}_2, \overline{\mathbf{e}}_3, \overline{\mathbf{e}}_4$
- Antiscalar unit (quadvector)

Homogeneous lines

- Take wedge product of two 4D points

$$\mathbf{P} = \left(P_x, P_y, P_z, 1\right) = P_x\mathbf{e}_1 + P_y\mathbf{e}_2 + P_z\mathbf{e}_3 + \mathbf{e}_4$$

$$\mathbf{Q} = \left(Q_x, Q_y, Q_z, 1\right) = Q_x\mathbf{e}_1 + Q_y\mathbf{e}_2 + Q_z\mathbf{e}_3 + \mathbf{e}_4$$

Homogeneous lines

$$\mathbf{P} \wedge \mathbf{Q} = (Q_x - P_x)\mathbf{e}_{41} + (Q_y - P_y)\mathbf{e}_{42} + (Q_z - P_z)\mathbf{e}_{43}$$
$$+ (P_yQ_z - P_zQ_y)\mathbf{e}_{23} + (P_zQ_x - P_xQ_z)\mathbf{e}_{31} + (P_xQ_y - P_yQ_x)\mathbf{e}_{12}$$

- This bivector spans a 2D plane in 4D
- In subspace where $w = 1$, this is a 3D line

Homogeneous lines

- The 4D bivector no longer contains any information about the two points used to create it
- Contrary to parametric origin / direction representation

Homogeneous lines

- The 4D bivector can be decomposed into two 3D components:
  - A tangent vector and a moment bivector
  - These are perpendicular

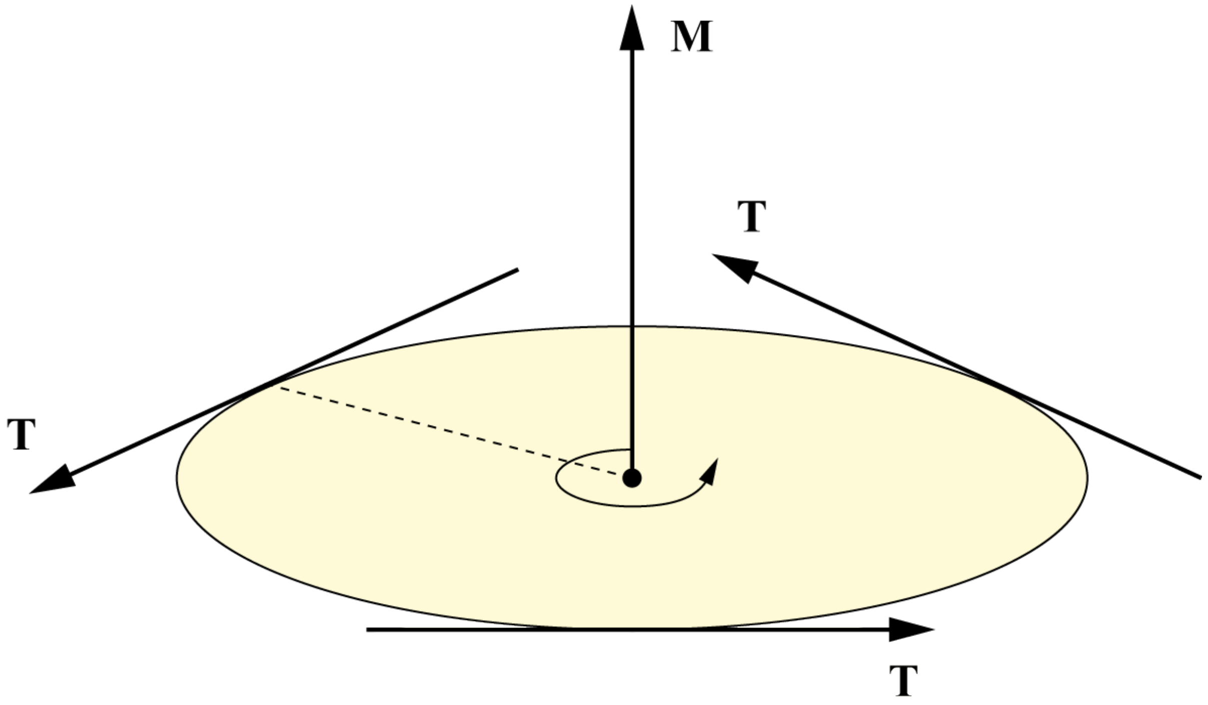
$$\mathbf{P} \wedge \mathbf{Q} = (Q_x - P_x)\mathbf{e}_{41} + (Q_y - P_y)\mathbf{e}_{42} + (Q_z - P_z)\mathbf{e}_{43}$$
$$+ (P_y Q_z - P_z Q_y)\mathbf{e}_{23} + (P_z Q_x - P_x Q_z)\mathbf{e}_{31} + (P_x Q_y - P_y Q_x)\mathbf{e}_{12}$$

Homogeneous lines

- Tangent **T** vector is $\mathbf{Q}_{3\mathrm{D}} - \mathbf{P}_{3\mathrm{D}}$
- Moment **M** bivector is $\mathbf{P}_{3\mathrm{D}} \wedge \mathbf{Q}_{3\mathrm{D}}$

$$\mathbf{P} \wedge \mathbf{Q} = (Q_x - P_x)\mathbf{e}_{41} + (Q_y - P_y)\mathbf{e}_{42} + (Q_z - P_z)\mathbf{e}_{43}$$
$$+ (P_y Q_z - P_z Q_y)\mathbf{e}_{23} + (P_z Q_x - P_x Q_z)\mathbf{e}_{31} + (P_x Q_y - P_y Q_x)\mathbf{e}_{12}$$

Moment bivector



Plücker coordinates

- Origin of Plücker coordinates revealed!
  - They are the coefficients of a 4D bivector
- A line **L** in Plücker coordinates is

$$\mathbf{L} = \{\mathbf{Q} - \mathbf{P} : \mathbf{P} \times \mathbf{Q}\}$$

- A bunch of seemingly arbitrary formulas in Plücker coordinates will become clear

Homogeneous planes

- Take wedge product of three 4D points

$$\mathbf{P} = (P_x, P_y, P_z, 1) = P_x\mathbf{e}_1 + P_y\mathbf{e}_2 + P_z\mathbf{e}_3 + \mathbf{e}_4$$

$$\mathbf{Q} = (Q_x, Q_y, Q_z, 1) = Q_x\mathbf{e}_1 + Q_y\mathbf{e}_2 + Q_z\mathbf{e}_3 + \mathbf{e}_4$$

$$\mathbf{R} = (R_x, R_y, R_z, 1) = R_x\mathbf{e}_1 + R_y\mathbf{e}_2 + R_z\mathbf{e}_3 + \mathbf{e}_4$$

Homogeneous planes

$$\mathbf{P} \wedge \mathbf{Q} \wedge \mathbf{R} = N_x \overline{\mathbf{e}}_1 + N_y \overline{\mathbf{e}}_2 + N_z \overline{\mathbf{e}}_3 + D\overline{\mathbf{e}}_4$$

- **N** is the 3D normal bivector
- *D* is the offset from origin in units of **N**

$$\mathbf{N} = \mathbf{P}_{3D} \wedge \mathbf{Q}_{3D} + \mathbf{Q}_{3D} \wedge \mathbf{R}_{3D} + \mathbf{R}_{3D} \wedge \mathbf{P}_{3D}$$

$$D = -\mathbf{P}_{3D} \wedge \mathbf{Q}_{3D} \wedge \mathbf{R}_{3D}$$

Plane transformation

- A homogeneous plane is a 4D antivector
- It transforms by the inverse of a 4 x 4 matrix
  - Just like a 3D antivector transforms by the inverse of a 3 x 3 matrix
  - Orthogonality not common here due to translation in the matrix

Projective geometry

4D Entity	3D Geometry
Vector (1-space)	Point (0-space)
Bivector (2-space)	Line (1-space)
Trivector (3-space)	Plane (2-space)

- We always project onto the 3D subspace where $w = 1$

Geometric computation in 4D

- Wedge product
  - Multiply **two points** to get the line containing both points
  - Multiply **three points** to get the plane containing all three points
  - Multiply **a line and a point** to get the plane containing the line and the point

Geometric computation in 4D

- Antiwedge product
  - Multiply **two planes** to get the line where they intersect
  - Multiply **three planes** to get the point common to all three planes
  - Multiply **a line and a plane** to get the point where the line intersects the plane

Geometric computation in 4D

- Wedge or antiwedge product
  - Multiply **a point and a plane** to get the signed minimum distance between them in units of the normal magnitude
  - Multiply **two lines** to get a special signed crossing value

Product of two lines

- Wedge product gives an antiscalar (quadvector or 4D volume element)
- Antiwedge product gives a scalar
- Both have same sign and magnitude
- Grassmann treated scalars and antiscalars as the same thing

Product of two lines

- Let $\mathbf{L}_1$ have tangent $\mathbf{T}_1$ and moment $\mathbf{M}_1$
- Let $\mathbf{L}_2$ have tangent $\mathbf{T}_2$ and moment $\mathbf{M}_2$
- Then,

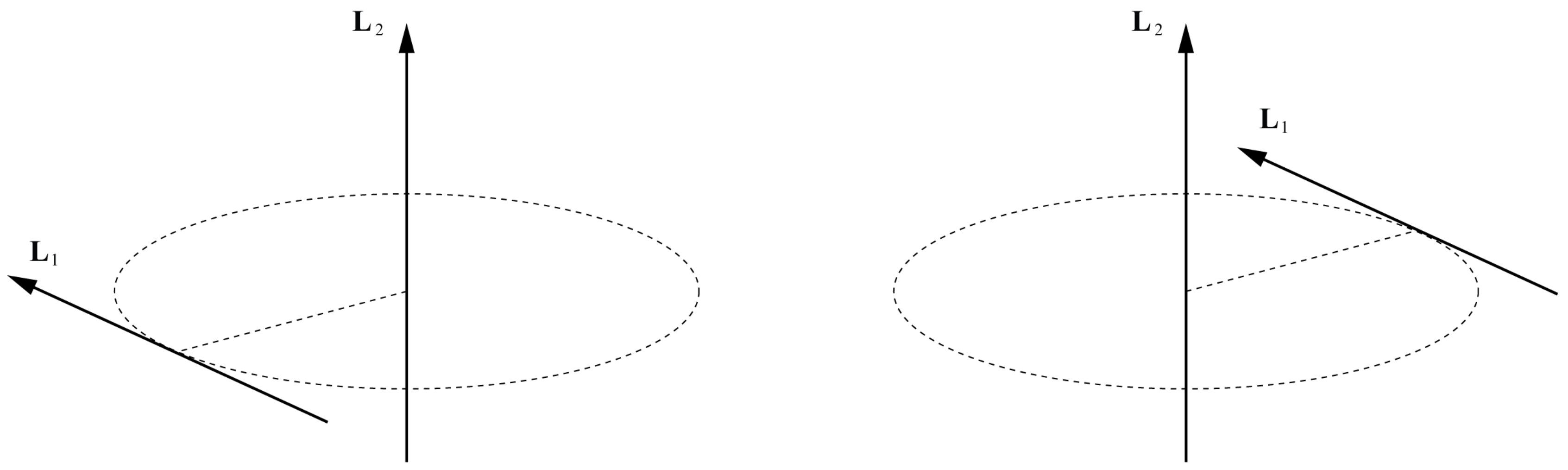
$$\mathbf{L}_1 \vee \mathbf{L}_2 = -(\mathbf{T}_1 \vee \mathbf{M}_2 + \mathbf{T}_2 \vee \mathbf{M}_1)$$

$$\mathbf{L}_1 \wedge \mathbf{L}_2 = -(\mathbf{T}_1 \wedge \mathbf{M}_2 + \mathbf{T}_2 \wedge \mathbf{M}_1)$$

Product of two lines

- The product of two lines gives a “crossing” relation
  - Positive value means clockwise crossing
  - Negative value means counterclockwise
  - Zero if lines intersect

Crossing relation



$\mathbf{L}_1 \vee \mathbf{L}_2 > 0$

$\mathbf{L}_1 \vee \mathbf{L}_2 < 0$

Distance between lines

- Product of two lines also relates to signed minimum distance between them

$$d = \frac{\mathbf{L}_1 \vee \mathbf{L}_2}{\left\| \mathbf{T}_1 \wedge \mathbf{T}_2 \right\|}$$

  - (Here, numerator is 4D antiwedge product, and denominator is 3D wedge product.)

Ray-triangle intersection

- Application of line-line product
- Classic barycentric calculation difficult due to floating-point round-off error
  - Along edge between two triangles, ray can miss both or hit both
  - Typical solution involves use of ugly epsilons

Ray-triangle intersection

- Calculate 4D bivectors for triangle edges and ray
  - Take antiwedge products between ray and three edges
  - Same sign for all three edges is a hit
  - Impossible to hit or miss both triangles sharing edge
  - Need to handle zero in consistent way

Weighting

- Points, lines, and planes have "weights" in homogeneous coordinates

Entity	Weight
Point	w coordinate
Line	Tangent component **T**
Plane	x, y, z component

Weighting

- Mathematically, the weight components can be found by taking the antiwedge product with the antivector (0,0,0,1)
- We would never really do that, though, because we can just look at the right coefficients

Normalized lines

- Tangent component has unit length
  - Magnitude of moment component is perpendicular distance to the origin

Normalized planes

- (x,y,z) component has unit length
  - Wedge product with (normalized) point is perpendicular distance to plane

Programming considerations

- Convenient to create classes to represent entities of each grade
  - Vector4D
  - Bivector4D
  - Antivector4D

Programming considerations

- Fortunate happenstance that C++ has an overloadable operator ^ that looks like a wedge
- But be careful with operator precedence if you overload ^ to perform wedge product
  - Has lowest operator precedence, so get used to enclosing wedge products in parentheses

Combining wedge and antiwedge

- The same operator can be used for wedge product and antiwedge product
  - Either they both produce the same scalar and antiscalar magnitudes with the same sign
  - Or one of the products is identically zero
  - For example, you would always want the antiwedge product for two planes because the wedge product is zero for all inputs

Summary

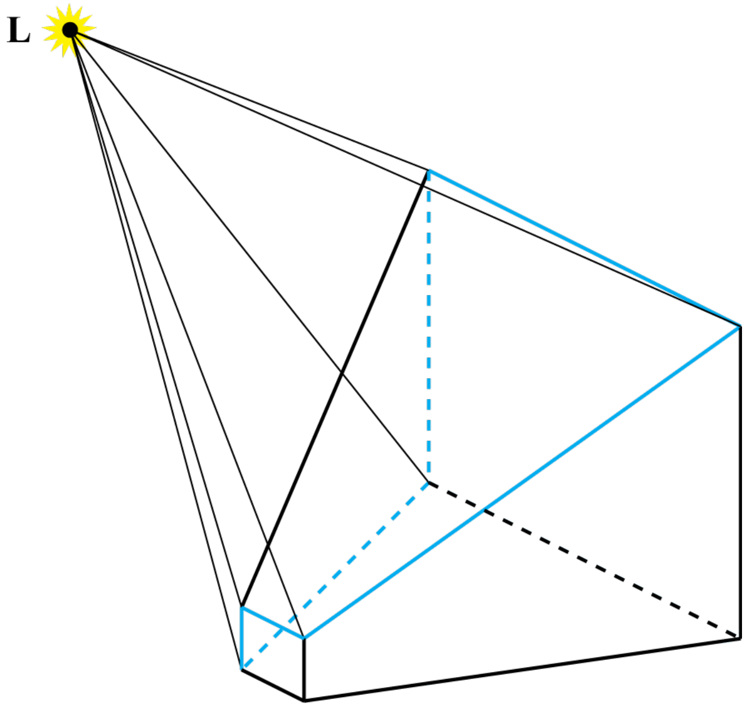
Old school	New school
Cross product → axial vector	Wedge product → bivector
Dot product	Antiwedge vector / antivector
Scalar triple product	Triple wedge product
Plücker coordinates	4D bivectors
Operations in Plücker coordinates	4D wedge / antiwedge products
Transform normals with inverse transpose	Transform antivectors with adjugate transpose

- Slides available online at
  - https://terathon.com/lengyel/
- Contact
  - lengyel@terathon.com

Supplemental Slides

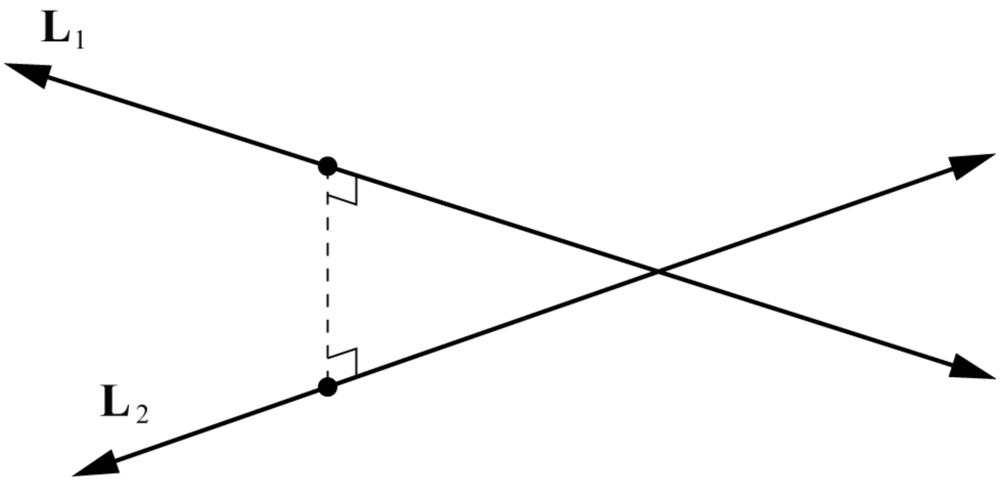
Example application

- Calculation of shadow region planes from light position and frustum edges
  - Simply a wedge product



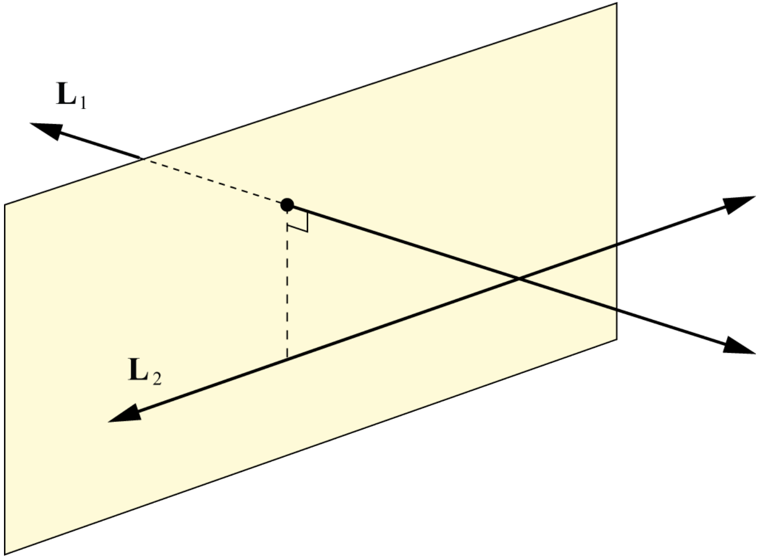
Points of closest approach

- Wedge product of line tangents gives complement of direction between closest points



Points of closest approach

- Plane containing this direction and first line also contains closest point on second line



Two dimensions

- 1 scalar unit
- 2 basis vectors
- 1 bivector / antiscalar unit
- No cross product
- All rotations occur in plane of 1 bivector

One dimension

- 1 scalar unit
- 1 single-component basis vector
  - Also antiscalar unit
- Equivalent to “dual numbers”
- All numbers have form $a + b\mathbf{e}$
  - Where $e^2 = 0$

Explicit formulas

- Define points **P**, **Q** and planes **E**, **F**, and line **L**

$$\mathbf{P} = \left(P_x, P_y, P_z, 1\right) = P_x\mathbf{e}_1 + P_y\mathbf{e}_2 + P_z\mathbf{e}_3 + \mathbf{e}_4$$

$$\mathbf{Q} = \left(Q_x, Q_y, Q_z, 1\right) = Q_x\mathbf{e}_1 + Q_y\mathbf{e}_2 + Q_z\mathbf{e}_3 + \mathbf{e}_4$$

$$\mathbf{E} = \left(E_x, E_y, E_z, E_w\right) = E_x\overline{\mathbf{e}}_1 + E_y\overline{\mathbf{e}}_2 + E_z\overline{\mathbf{e}}_3 + E_w\overline{\mathbf{e}}_4$$

$$\mathbf{F} = \left(F_x, F_y, F_z, F_w\right) = F_x\overline{\mathbf{e}}_1 + F_y\overline{\mathbf{e}}_2 + F_z\overline{\mathbf{e}}_3 + F_w\overline{\mathbf{e}}_4$$

$$\mathbf{L} = T_x\mathbf{e}_{41} + T_y\mathbf{e}_{42} + T_z\mathbf{e}_{43} + M_x\mathbf{e}_{23} + M_y\mathbf{e}_{31} + M_z\mathbf{e}_{12}$$

Explicit formulas

- Product of two points

$$\mathbf{P} \wedge \mathbf{Q} = (Q_x - P_x)\mathbf{e}_{41} + (Q_y - P_y)\mathbf{e}_{42} + (Q_z - P_z)\mathbf{e}_{43}$$
$$+ (P_y Q_z - P_z Q_y)\mathbf{e}_{23} + (P_z Q_x - P_x Q_z)\mathbf{e}_{31} + (P_x Q_y - P_y Q_x)\mathbf{e}_{12}$$

Explicit formulas

- Product of two planes

$$\mathbf{E} \vee \mathbf{F} = (E_z F_y - E_y F_z)\mathbf{e}_{41} + (E_x F_z - E_z F_x)\mathbf{e}_{42} + (E_y F_x - E_x F_y)\mathbf{e}_{43}$$
$$+ (E_x F_w - E_w F_x)\mathbf{e}_{23} + (E_y F_w - E_w F_y)\mathbf{e}_{31} + (E_z F_w - E_w F_z)\mathbf{e}_{12}$$

Explicit formulas

- Product of line and point

$$\mathbf{L} \wedge \mathbf{P} = (T_y P_z - T_z P_y + M_x)\overline{\mathbf{e}}_1 + (T_z P_x - T_x P_z + M_y)\overline{\mathbf{e}}_2 + (T_x P_y - T_y P_x + M_z)\overline{\mathbf{e}}_3 + (-P_x M_x - P_y M_y - P_z M_z)\overline{\mathbf{e}}_4$$

Explicit formulas

- Product of line and plane

$$\mathbf{L} \vee \mathbf{E} = (M_zE_y - M_yE_z - T_xE_w)\mathbf{e}_1 + (M_xE_z - M_zE_x - T_yE_w)\mathbf{e}_2 \\ + (M_yE_x - M_xE_y - T_zE_w)\mathbf{e}_3 + (E_xT_x + E_yT_y + E_zT_z)\mathbf{e}_4$$